Far-field Pressure due to a Planar Piston of Arbitrary Shape

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In this paper I describe a technique for calculating the far-field pressure due to a vibrating piston of arbitrary shape located on an infinite rigid planar baffle. The technique is based on a two-dimensional version of Green's theorem. Integral expressions over the area of the piston are converted to line integrals around the perimeter of the piston. Explicit relations are derived for polygonal pistons with any number of sides.

Let P be denote a region in a plane infinite rigid baffle that is udergoing piston-like vibration with normal displacement u and let o be an origin located on P. Using the far-field representation of the infinite plane Green's function, the pressure at a far-field point x due to this piston is given by

$$p(x) = -\frac{\omega^2 \rho u}{2\pi} \frac{e^{-ik|x-o|}}{|x-o|} \int_P e^{ik(x-o)\cdot(y-o)/|x-o|}\, dS(y) \tag{1}$$

where ω is the angular frequency, ρ is the fluid density, k is the acoustic wave number ($k = \omega/c$), and c is the fluid sound speed. Define

$$I(x) = \int_P e^{ik(x-o)\cdot(y-o)/|x-o|}\, dS(y). \tag{2}$$

Then, it follows from equation (1) that

$$p(x) = -\frac{\omega^2 \rho u}{2\pi} \frac{e^{-ik|x-o|}}{|x-o|} I(x). \tag{3}$$

If $x_0 - o$ is the projection of $(x-o)/|x-o|$ onto the plane of P, then

$$I(x) = \int_P e^{ik(x_0-o)\cdot(y-o)}\, dS(y). \tag{4}$$

Green's theorem in two dimensions can be written

$$\int_P \nabla\phi = \oint_{\partial P} \phi\vec{n} \tag{5}$$

where ∇ is the two-dimensional gradient in the plane of P, ∂P is the boundary of P, and $\vec{n}$ is the outward unit normal to ∂P. If we choose ϕ to be the function

$$\phi(y) = e^{ik(x_0-o)\cdot(y-o)}, \tag{6}$$

then

$$\nabla\phi(y) = ik(x_0 - o)e^{ik(x_0-o)\cdot(y-o)} \tag{7}$$

and hence

$$\begin{aligned}\int_P \nabla\phi &= ik(x_0 - o)\int_P e^{ik(x_0-o)\cdot(y-o)}\,dS(y)\\ &= ik(x_0-o)I\\ &= \oint_{\partial P}\phi\vec{n}\\ &= \oint_{\partial P} e^{ik(x_0-o)\cdot(y-o)}\,\vec{n}\,ds(y).\end{aligned} \tag{8}$$

If $x_0 = o$, then it follows from equation (4) that $I(x) = \text{Area}(P)$. Assume that $x_0 \neq o$. Taking the dot product of equation (8) with $x_0 - o$, we get

$$I(x) = \frac{1}{ik(x_0-o)\cdot(x_0-o)}\oint_{\partial P}\big((x_0-o)\cdot\vec{n}\big)e^{ik(x_0-o)\cdot(y-o)}\,ds(y). \tag{9}$$

This equation together with equation (3) provides a general expression for the far-field pressure in terms of a line integral around the perimeter. If $x - o$ is written in terms of a spherical coordinate system with origen at o and z-axis perpendicular to the plane of the piston, $x_0 - o$ depends only on the angular coordinates θ and ϕ.

Suppose now that P is a polygonal region. Let E_m denote the m-th edge of the polygon P and let $\vec{n}_m$ denote the outward unit normal to E_m in the plane. Then it follows from equation (9) that

$$I(x) = \frac{1}{ik(x_0-o)\cdot(x_0-o)}\sum_{m=1}^{M}\big((x_0-o)\cdot\vec{n}_m\big)\oint_{E_m} e^{ik(x_0-o)\cdot(y-o)}\,ds(y) \tag{10}$$

where M is the number of edges of P. The edge E_m can be described by the parametric equation

$$y(\tau) = \bar{x}_m + \tau\vec{t}_m \qquad -s_m/2 \leq \tau \leq s_m/2 \tag{11}$$

where $\bar{x}_m$ is the mid-point of the m-th edge, $\vec{t}_m$ is a unit tangent vector along the m-th edge, and s_m is the length of the m-th edge. The integral in equation (10) can now be written

$$\oint_{E_m} e^{ik(x_0-o)\cdot(y-o)}\, ds(y) = \int_{-s_m/2}^{s_m/2} e^{ik(x_0-o)\cdot(\bar{x}_m - o + \tau \vec{t}_m)}\, d\tau$$
$$= \frac{2e^{ik(x_0-o)\cdot(\bar{x}_m-o)}}{k(x_0-o)\cdot\vec{t}_m} \sin\left(k\frac{s_m}{2}(x_0-o)\cdot\vec{t}_m\right). \tag{12}$$

Combining equations (10) and (12), we get

$$I(x) = \frac{2}{ik^2(x_0-o)\cdot(x_0-o)} \sum_{m=1}^{M} \left(\frac{(x_0-o)\cdot\vec{n}_m}{(x_0-o)\cdot\vec{t}_m}\right) e^{ik(x_0-o)\cdot(\bar{x}_m-o)} \cdot \sin\left(k\frac{s_m}{2}(x_0-o)\cdot\vec{t}_m\right). \tag{13}$$

It folows from equation (3) and equation (13) that

$$p(x) = \frac{-\rho c^2 u}{i\pi(x_0-o)\cdot(x_0-o)} \frac{e^{-ik|x-o|}}{|x-o|} \sum_{m=1}^{M} \left(\frac{(x_0-o)\cdot\vec{n}_m}{(x_0-o)\cdot\vec{t}_m}\right) e^{ik(x_0-o)\cdot(\bar{x}_m-o)} \cdot \sin\left(k\frac{s_m}{2}(x_0-o)\cdot\vec{t}_m\right). \tag{14}$$