



Mutual Interaction of Pistons of Arbitrary Shape on a Planar Rigid Baffle

Dr. George W Benthien

November 1985

E-mail: george@gbenthien.net

Oscar Lindeman in a 1974 JASA article [Lindeman, Oscar, *Transient fluid reaction on a baffled plane piston of arbitrary shape*, J. Acoust. Soc. Am., **55**, No. 4, April 1974] presented a technique for obtaining the self radiation impedance of a baffled piston on a plane as a Fourier transform of its impulse response. The impulse response was expressed as a double contour integral around the pistons perimeter. This technique has been applied by a number of authors to pistons of various shapes. In this paper I will illustrate how Lindeman's technique can be extended to the mutual acoustic interaction between two pistons of arbitrary shape on a rigid planar baffle. We will denote the two pistons by \mathcal{P} and \mathcal{P}' as shown in figure 1.

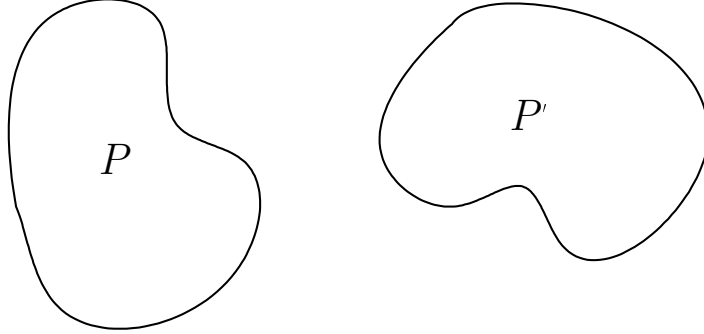


Figure 1: Two planar pistons of arbitrary shape on an infinite rigid baffle

The mutual radiation impedance \mathcal{Z} is defined by

$$\mathcal{Z} = \rho c \frac{ik}{2\pi} \int_P \int_{P'} \frac{e^{-ik|x-x'|}}{|x-x'|} dS(x)dS(x'). \quad (1)$$

We will show that the double surface integral in equation (1) can be reduced to a simpler expression involving only line integrals. The technique used is similar to that used by Lindeman to calculate self radiation reactions [Lindeman, Oscar, *Transformation of the Helmholtz integral into a Line Integral*, J. Acoust. Soc. Am., **40**, no. 4, 1966]. It relies on the two dimensional form of Green's theorem.

By Green's theorem in two dimensions

$$\int_P \nabla_x \phi(x, x') dS(x) = \oint_{\partial P} \phi(x, x') \vec{n}(x) ds(x) \quad (2)$$

where ∂P is the boundary of P and \vec{n} is the outward unit normal to ∂P . By the divergence theorem in two dimensions

$$\int_{P'} \text{div } \vec{u}(x') dS(x') = \oint_{\partial P'} \vec{u}(x') \cdot \vec{n}(x') ds(x'). \quad (3)$$

It follows from equations (2) and (3) that

$$\begin{aligned} \int_{P'} \operatorname{div}_{x'} \left[\int_P \nabla_x \phi(x, x') dS(x) \right] dS(x') &= \oint_{\partial P'} \vec{n}(x') \cdot \left[\int_P \nabla_x \phi(x, x') dS(x) \right] dS(x') \\ &= \oint_{\partial P'} \vec{n}(x') \cdot \left[\oint_{\partial P} \phi(x, x') \vec{n}(x) ds(x) \right] dS(x'). \end{aligned} \quad (4)$$

Suppose ϕ has the form

$$\phi(x, x') = \Phi(x - x'). \quad (5)$$

Then

$$\nabla_x \phi(x, x') = -\nabla_{x'} \phi(x, x') \quad (6)$$

and

$$\Delta_x \phi(x, x') = \Delta_{x'} \phi(x, x'). \quad (7)$$

Combining equations (4)–(7), we get

$$\begin{aligned} \oint_{\partial P'} \vec{n}(x') \cdot \left[\oint_{\partial P} \phi(x, x') \vec{n}(x) ds(x) \right] dS(x') &= - \int_{P'} \operatorname{div}_{x'} \left[\int_P \nabla_{x'} \phi(x, x') dS(x) \right] dS(x') \\ &= \int_{P'} \left[\int_P \Delta_{x'} \phi(x, x') dS(x) \right] dS(x') \\ &= \int_{P'} \left[\int_P \Delta_x \phi(x, x') dS(x) \right] dS(x'). \end{aligned} \quad (8)$$

If ϕ has the form

$$\phi(x, x') = \psi(r) \quad \text{with } r = |x - x'|, \quad (9)$$

then

$$\Delta_x \phi(x, x') = \frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right). \quad (10)$$

Combining equations (8) and (10), we get

$$\oint_{\partial P'} \vec{n}(x') \cdot \left[\oint_{\partial P} \phi(x, x') \vec{n}(x) ds(x) \right] dS(x') = - \int_{P'} dS(x') \int_P \frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) dS(x). \quad (11)$$

If we define

$$\psi(r) = \int_0^r \frac{1 - e^{-ikz}}{z} dz, \quad (12)$$

then

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) = ik \frac{e^{-ikr}}{r}. \quad (13)$$

Combining equations (11)–(13), we get

$$\begin{aligned}
ik \int_{P'} dS(x') \int_P \frac{e^{-ik|x-x'|}}{|x-x'|} dS(x) &= - \oint_{\partial P'} \vec{n}(x') \cdot \left[\oint_{\partial P} \psi(|x-x'|) \vec{n}(x) ds(x) \right] ds(x') \\
&= \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \int_0^{|x-x'|} \frac{1-e^{-ikz}}{z} dz ds(x) ds(x').
\end{aligned} \tag{14}$$

If we define $\chi(x, x', z)$ by

$$\chi(x, x', z) = \begin{cases} 1 & |x-x'| \geq z \\ 0 & |x-x'| < z \end{cases} \tag{15}$$

then

$$\int_0^{|x-x'|} \frac{1-e^{-ikz}}{z} dz = \int_0^w \chi(x, x', z) \frac{1-e^{-ikz}}{z} dz \tag{16}$$

where

$$w = \max_{\substack{x \in P \\ x' \in P'}} |x-x'|. \tag{17}$$

Substituting equation (16) into equation (14), we get

$$\begin{aligned}
ik \int_{P'} dS(x') \int_P \frac{e^{-ik|x-x'|}}{|x-x'|} dS(x) &= - \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \int_0^w \chi(x, x', z) \frac{1-e^{-ikz}}{z} dz \\
&= 2\pi \int_0^w f(z) (1-e^{-ikz}) dz
\end{aligned} \tag{18}$$

where

$$f(z) = -\frac{1}{2\pi z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \chi(x, x', z) ds(x) ds(x'). \tag{19}$$

Substituting equation (18) into equation (1), we get

$$\mathcal{Z} = \rho c \int_0^w f(z) (1-e^{-ikz}) dz. \tag{20}$$

It is shown in the appendix that

$$\int_0^w f(z) dz = \text{Area}(P \cap P'). \tag{21}$$

Combining equations (20) and (21), we get

$$\boxed{\mathcal{Z} = \rho c \text{Area}(P \cap P') - \rho c \int_0^w f(z) e^{-ikz} dz.} \tag{22}$$

Letting $t = z/c$ in the above integral, we obtain

$$\mathcal{Z}(\omega) = \rho c \text{Area}(P \cap P') - \rho c^2 \int_0^{w/c} f(ct) e^{-i\omega t} dt. \quad (23)$$

The radiation impulse response $\zeta(t)$ of the piston is the inverse Fourier transform of $\mathcal{Z}(\omega)$, i.e.,

$$\zeta(t) = \rho c \text{Area}(P \cap P') \delta(t) - \rho c^2 f(ct) \quad (24)$$

where $\delta(t)$ is the Dirac delta function.

Since $\oint_{\partial P} \vec{n}(x) ds(x) = 0$, f can be written in the alternate form

$$\begin{aligned} f(z) &= \frac{1}{2\pi z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) [1 - \chi(x, x', z)] ds(x) ds(x') \\ &= \frac{1}{2\pi z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \hat{\chi}(x, x', z) ds(x) ds(x') \end{aligned} \quad (25)$$

where

$$\hat{\chi}(x, x', z) = 1 - \chi(x, x', z) = \begin{cases} 1 & |x - x'| < z \\ 0 & |x - x'| \geq z \end{cases}. \quad (26)$$

If we define

$$g(z, x') = \oint_{\partial P} \vec{n}(x') \cdot \vec{n}(x) \hat{\chi}(x, x', z) ds(x), \quad (27)$$

then

$$f(z) = \frac{1}{2\pi z} \oint_{\partial P'} g(z, x') ds(x'). \quad (28)$$

Geometrically, $g(z, x')$ is the signed projection of the portion of ∂P within a circle of radius z centered at x' onto the tangent line at x' . For example, $g(z, x')$ is the sum of the lengths of the segments ab and cd in figure 2 for the case $P = P'$. There is no contribution in the interval (b, c) since the contributions of the two portions of ∂P within the circle have opposite signs and cancel.

It is clear from the figure that, for small z , $g(z, x')$ is approximately $2z$. This is true for all points x' where ∂P is smooth. We will assume that the set of points where ∂P is not smooth is a set of measure zero. Therefore,

$$f(0^+) = \sigma/\pi \quad \text{when} \quad P = P' \quad (29)$$

where σ is the perimeter of P . When $P \cap P' = \emptyset$, there is a minimum distance between points on ∂P and points on $\partial P'$. If z is less than this minimum distance, then $g(z, x') = 0$ for all $x' \in \partial P'$. Therefore,

$$f(0^+) = 0 \quad \text{when} \quad P \cap P' = \emptyset. \quad (30)$$

It is also clear from the geometric interpretation of $g(z, x')$ that it is bounded by $2z$. Therefore, $f(z)$ is bounded by σ/π .

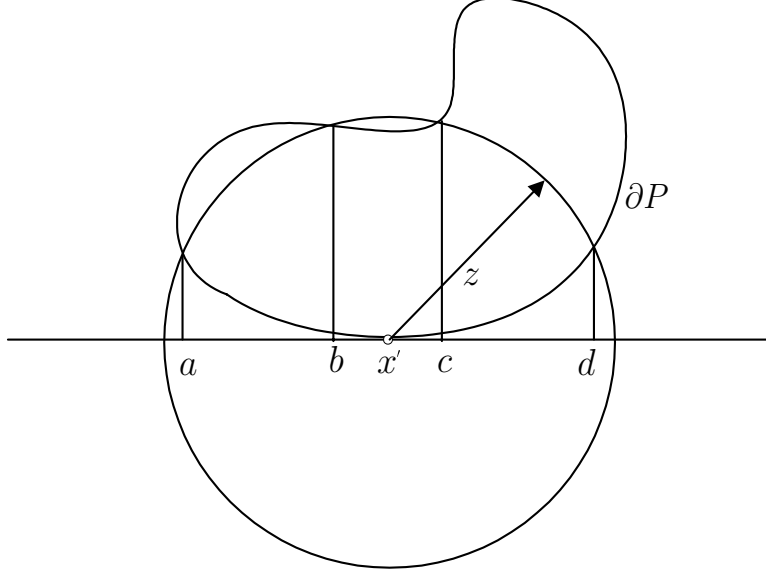


Figure 2: Geometric interpretation of $g(z, x')$ for the case $P = P'$

Appendix

In this appendix we want to show that

$$\int_0^w f(z) dz = \text{Area}(P \cap P').$$

It was shown in equation (25) that

$$f(z) = \frac{1}{2\pi z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \hat{\chi}(x, x', z) ds(x) ds(x').$$

Therefore,

$$\begin{aligned} \int_0^w f(z) dz &= \frac{1}{2\pi} \int_0^w \frac{dz}{z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \hat{\chi}(x, x', z) ds(x) ds(x') \\ &= \frac{1}{2\pi} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \int_{|x-x'|}^w \frac{dz}{z} ds(x) ds(x') \\ &= \frac{1}{2\pi} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \ln|x-x'| ds(x) ds(x'). \end{aligned} \quad (31)$$

In deriving equation (31) we used the fact that

$$\oint_{\partial P} \vec{n}(x) ds(x) = 0.$$

If $x' \notin P$, then it follows from Green's theorem that

$$\oint_{\partial P} \vec{n} \ln |x - x'| ds(x) = \int_P \nabla_x \ln |x - x'| dS(x). \quad (32)$$

Suppose $x' \in P$. Let $S_\epsilon(x')$ be a circle of radius ϵ centered at x' [see figure 3]. By Green's

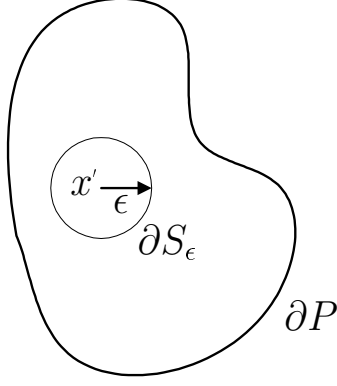


Figure 3: Isolation of singular point by a circle of radius ϵ

theorem

$$\oint_{\partial P + \partial S_\epsilon(x')} \vec{n} \ln |x - x'| ds(x) = \int_{P - S_\epsilon(x')} \nabla_x \ln |x - x'| dS(x). \quad (33)$$

But

$$\oint_{\partial S_\epsilon(x')} \vec{n} \ln |x - x'| ds(x) = \ln \epsilon \oint_{\partial S_\epsilon(x')} \vec{n} ds(x) = 0. \quad (34)$$

Thus,

$$\oint_{\partial P} \vec{n} \ln |x - x'| ds(x) = \lim_{\epsilon \rightarrow 0} \int_{P - S_\epsilon(x')} \nabla_x \ln |x - x'| dS(x) = \int_P \nabla_x \ln |x - x'| dS(x). \quad (35)$$

Combining equations (31)–(35), we get

$$\begin{aligned} \int_0^w f(z) dz &= \frac{1}{2\pi} \oint_{\partial P'} \vec{n}(x') \cdot \int_P \nabla_x \ln |x - x'| dS(x) ds(x') \\ &= -\frac{1}{2\pi} \oint_{\partial P'} \vec{n}(x') \cdot \int_P \nabla_{x'} \ln |x - x'| dS(x) ds(x') \\ &= -\frac{1}{2\pi} \int_P dS(x) \oint_{\partial P'} \nabla_{x'} \ln |x - x'| \cdot \vec{n}(x') ds(x'). \end{aligned} \quad (36)$$

If $x \notin P'$, then it follows from the divergence theorem that

$$\begin{aligned} \oint_{\partial P'} \nabla_{x'} \ln |x - x'| \cdot \vec{n}(x') ds(x') &= \int_{P'} \Delta_{x'} \ln |x - x'| dS(x') \\ &= 0. \end{aligned} \quad (37)$$

Thus,

$$\int_0^w f(z) dz = -\frac{1}{2\pi} \int_{P \cap P'} dS(x) \oint_{\partial P'} \nabla_{x'} \ln |x - x'| \cdot \vec{n}(x') ds(x'). \quad (38)$$

Let $S_\epsilon(x)$ be a circle around x of radius ϵ . Then, it follows from the divergence theorem that

$$\begin{aligned} \oint_{\partial P' + \partial S_\epsilon} \nabla_{x'} \ln |x - x'| \cdot \vec{n}(x') ds(x') &= \int_{P' - S_\epsilon} \Delta_{x'} \ln |x - x'| dS(x') \\ &= 0 \end{aligned} \quad (39)$$

or

$$\oint_{\partial P'} \nabla_{x'} \ln |x - x'| \cdot \vec{n}(x') ds(x') = - \oint_{\partial S_\epsilon} \nabla_{x'} \ln |x - x'| \cdot \vec{n}(x') ds(x'). \quad (40)$$

However,

$$\begin{aligned} \oint_{\partial S_\epsilon} \nabla_{x'} \ln |x - x'| \cdot \vec{n}(x') ds(x') &= \oint_{\partial S_\epsilon} \frac{1}{|x - x'|} \nabla_{x'} |x - x'| \cdot \vec{n}(x') ds(x') \\ &= \oint_{\partial S_\epsilon} \frac{1}{|x - x'|} \frac{x' - x}{|x - x'|} \cdot \vec{n}(x') ds(x') \\ &= \oint_{\partial S_\epsilon} \frac{1}{|x - x'|} ds(x') \\ &= \frac{1}{\epsilon} \oint_{\partial S_\epsilon} ds(x') \\ &= 2\pi. \end{aligned} \quad (41)$$

Combining equations (40) and (41), we get

$$\oint_{\partial P'} \nabla_{x'} \ln |x - x'| \cdot \vec{n}(x') ds(x') = -2\pi. \quad (42)$$

Substituting equation (42) into equation (38), we get

$$\int_0^w f(z) dz = \int_{P \cap P'} dS(x) = \text{Area}(P \cap P'). \quad (43)$$

In particular, we have

$$\int_0^w f(z) dz = \begin{cases} \text{Area}(P) & \text{for } P = P' \\ 0 & \text{for } P \cap P' = \emptyset \end{cases}. \quad (44)$$