



Symmetry Reductions

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1 Introduction

The numerical approximation of many physical problems leads to the solution of a system of linear algebraic equations of the form

$$Ax = b \tag{1}$$

where A is an $N \times N$ matrix representing some linear operator, b is an N -vector representing a forcing function, and x is an N -vector of unknowns corresponding to the values of some scalar physical quantity at N locations in a spatial region of interest. For example, the components of x might correspond to the values of pressure or electric potential at a discrete set of points. In this paper we have restricted our attention primarily to problems involving scalar physical variables, but the general techniques developed can be extended to problems involving vector variables by appropriately extending the definition of the symmetry operators. Symmetry in the physical problem manifests itself in the structure of the matrix A . For example, with proper numbering of the evaluation points, one plane of symmetry leads to the following structure of A

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}. \tag{2}$$

Here A_1 and A_2 are submatrices of A that are one-half the size of A . The special forms resulting from various types of symmetry can be utilized to significantly reduce the solution time of equation (1). In this paper we will consider the following types of symmetry

1. one, two, or three planes of symmetry
2. any finite order of rotational symmetry
3. any finite order of rotational symmetry plus one additional plane of symmetry.

It is not necessary for the forcing vector b to have the same symmetry as the rest of the problem in order to achieve reductions in solution times, but additional reductions can be achieved if the forcing vector has the same symmetry. Table 1 shows the reductions in computational times that can be obtained by taking advantage of these symmetries.

2 Basic Theory

In this section we will discuss the basic theory behind the symmetry reduction schemes that are the subject of this paper. All of the symmetry reductions we will consider are based on two simple concepts. The first is that the symmetry in the physical problem manifests itself in the commutation of the operator A in equation (3) with certain symmetry operators (reflections and/or rotations). The second is a result from linear algebra stating that the eigenspaces of an operator are invariant under any operator that commutes with it. In the remainder of this section we will apply these concepts to the various types of symmetry under consideration.

Table 1: Time reduction factors for various types of symmetry

Symmetry Type	Solution	Solution with right-hand-side symmetry	Matrix generation
1 plane of symmetry	4	8	2
2 planes of symmetry	16	64	4
3 planes of symmetry	64	512	8
N -fold rotational symmetry	N^2	N^3	N
N -fold rotational symmetry plus 1 plane of symmetry	$4N^2$	$8N^3$	$2N$

2.1 One plane of symmetry

Consider a planar region as shown in figure 1 that has one plane of symmetry. The evaluation points are numbered from 1 to 16.

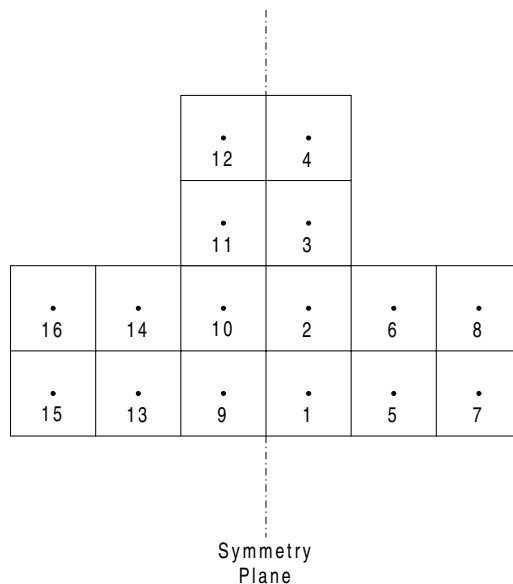


Figure 1: Subdivided region with one plane of symmetry

Notice that the evaluation points are symmetrically located relative to the symmetry plane and that the numbering of points in symmetric portions of the region is in the same order. This symmetric numbering greatly simplifies the resulting matrix structure. Suppose that

the physics of the problem leads to a system of linear equations

$$Ax = b \tag{3}$$

for the unknown vector x , where the components of x correspond to the values of some scalar physical variable at the evaluation points shown. Let x be partitioned into two parts corresponding to the two sides of the symmetry plane as follows

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{4}$$

The operator Σ corresponding to reflection across the symmetry plane is defined by

$$\Sigma x = \Sigma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \quad \text{for all } x. \tag{5}$$

It is easily verified that Σ must have the block form

$$\Sigma = \begin{pmatrix} 0 & \text{I} \\ \text{I} & 0 \end{pmatrix} \tag{6}$$

where I is an identity matrix. Suppose that the forcing vector b is reflected across the symmetry plane, i.e., b is replaced by Σb . The symmetry of the problem implies that the resulting solution x would also be reflected across the symmetry plane, i.e., x becomes Σx . For example, the response at location 11 due to a unit forcing function at location 4 is the same as the response at location 3 due to a unit forcing function at location 12. Thus, in general

$$A\Sigma x = \Sigma b = \Sigma Ax. \tag{7}$$

Since equation (7) must hold for any choice of b (and hence x), it follows that

$$A\Sigma = \Sigma A, \tag{8}$$

i.e., the operators Σ and A commute. Substituting the partitioned forms of A and Σ into equation (8) and carrying out the block multiplications shows that A has the block form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}. \tag{9}$$

Commuting operators have the property that the eigenspaces of one of the operators are invariant under the other operator. For example, if e is an eigenvector of Σ corresponding to the eigenvalue λ , then

$$\Sigma(Ae) = A(\Sigma e) = \lambda(Ae), \tag{10}$$

i.e., Ae is also an eigenvector of Σ corresponding to the eigenvalue λ . Since symmetry operators like Σ have relatively simple eigenstructures that can be determined by inspection or simple analysis, we will use the fact that these eigenspaces are invariant under A . The eigenvalues of Σ are ± 1 . A basis of eigenvectors corresponding to the eigenvalue $+1$ is given

by the columns of the matrix $\begin{pmatrix} \mathbf{I} \\ \mathbf{I} \end{pmatrix}$ where \mathbf{I} is the identity matrix. Similarly, a basis of eigenvectors corresponding to -1 is given by $\begin{pmatrix} \mathbf{I} \\ -\mathbf{I} \end{pmatrix}$. If we change to the basis given by the columns of

$$S = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix} \quad (11)$$

then the invariance of the eigenspaces of Σ under A implies that the matrix will become block diagonal relative to this basis. Under this change of basis A becomes $S^{-1}AS$. The matrix S has the property

$$S^{-1} = \frac{1}{2}S. \quad (12)$$

Thus,

$$S^{-1}AS = \frac{1}{2}SAS = \begin{pmatrix} A_1 + A_2 & 0 \\ 0 & A_1 - A_2 \end{pmatrix} \quad (13)$$

and the system of equations (3) becomes

$$(S^{-1}AS)(S^{-1}x) = \begin{pmatrix} A_1 + A_2 & 0 \\ 0 & A_1 - A_2 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = S^{-1}b = \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} \quad (14)$$

where

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = S^{-1}x. \quad (15)$$

Equations (14)–(15) imply that the original system of equations (3) can be replaced by two systems of half the size, i.e.,

$$\begin{aligned} (A_1 + A_2)\hat{x}_1 &= \hat{b}_1 \\ (A_1 - A_2)\hat{x}_2 &= \hat{b}_2 \end{aligned} \quad (16a)$$

where

$$\begin{aligned} \hat{b}_1 &= \frac{1}{2}(b_1 + b_2) \\ \hat{b}_2 &= \frac{1}{2}(b_1 - b_2) \end{aligned} \quad (16b)$$

and

$$\begin{aligned} x_1 &= \hat{x}_1 + \hat{x}_2 \\ x_2 &= \hat{x}_1 - \hat{x}_2. \end{aligned} \quad (16c)$$

Since solution time is proportional to the cube of the number of equations, it is four times faster to solve the two smaller systems than the original larger system. If the right-hand-side b is symmetric across the symmetry plane, then it follows that $b_1 = b_2$, $\hat{b}_1 = b_1$, $\hat{b}_2 = 0$, $\hat{x}_2 = 0$, and $x_1 = x_2 = \hat{x}_1$. In this case it is only necessary to solve the single system of equations

$$(A_1 + A_2)x_1 = b_1 \quad (17)$$

and set $x_2 = x_1$.

2.2 Some Extensions of the theory

We have made some simplifying assumptions in order to make the explanations easier. For example, we have assumed that the underlying physical variable is scalar and that none of the points of evaluation lie on a symmetry plane. To extend the procedure to vector problems, the components of x and b are replaced by 3-vectors. The symmetry operator Σ has the same form as in the scalar case with the ones in the identity matrices replaced by the 3-vector reflection operator σ . With these changes the basic procedure can be applied as before.

Points lying on the symmetry plane can be handled in at least two ways. In the first way the vectors x and b are further partitioned so that the points lying on the symmetry plane are grouped together. For example, with one symmetry plane the vector x can be partitioned as follows

$$x = \begin{pmatrix} x_+ \\ x_0 \\ x_- \end{pmatrix} \quad (18)$$

where x_0 corresponds to points on the symmetry plane. The symmetry operator Σ now becomes

$$\Sigma = \begin{pmatrix} 0 & 0 & \text{I} \\ 0 & \text{I} & 0 \\ \text{I} & 0 & 0 \end{pmatrix}. \quad (19)$$

The commutation of A with Σ implies that A must have the form

$$A = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_4 \\ A_3 & A_2 & A_1 \end{pmatrix}. \quad (20)$$

As before, the eigenvalues are ± 1 , but now the eigenspace corresponding to $+1$ is of higher dimension than the eigenspace corresponding to -1 . The eigenmatrix S for this case is given by

$$S = \begin{pmatrix} \text{I} & 0 & \text{I} \\ 0 & \text{I} & 0 \\ \text{I} & 0 & -\text{I} \end{pmatrix}. \quad (21)$$

Applying the change of basis corresponding to S gives

$$S^{-1}AS = \begin{pmatrix} A_1 + A_3 & A_2 & 0 \\ 2A_4 & A_5 & 0 \\ 0 & 0 & A_1 - A_3 \end{pmatrix}. \quad (22)$$

The case with points on the symmetry plane can also be handled by double numbering the points on the symmetry plane.

2.3 Two planes of symmetry

Consider next a problem having two planes of symmetry. The four symmetry quadrants will be numbered as shown in table 2. Two planes of symmetry can be handled by applying

Table 2: Numbering of quadrants for two planes of symmetry

Quadrant	x_1	x_2
1	+	+
2	+	-
3	-	+
4	-	-

the results for one plane of symmetry successively to the two reflection operators. If the evaluation points are numbered symmetrically in the four symmetry quadrants, then the two reflection operators Σ_1 and Σ_2 can be written

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \end{pmatrix} \quad \Sigma_2 = \begin{pmatrix} 0 & \mathbf{I} & 0 & 0 \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I} \\ 0 & 0 & \mathbf{I} & 0 \end{pmatrix}. \quad (23)$$

Notice that

$$\Sigma_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \Sigma_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{pmatrix}. \quad (24)$$

The fact that A commutes with both Σ_1 and Σ_2 implies that A must have the block structure

$$A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ A_3 & A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix}. \quad (25)$$

The symmetry operators Σ_1 and Σ_2 also commute with each other. Thus, each eigenspace of Σ_1 can be further subdivided into eigenspaces of the operator Σ_2 restricted to this eigenspace.

The eigenvector matrices S_1 and S_2 corresponding to Σ_1 and Σ_2 are

$$S_1 = \begin{pmatrix} \mathbf{I} & 0 & \mathbf{I} & 0 \\ 0 & \mathbf{I} & 0 & \mathbf{I} \\ \mathbf{I} & 0 & -\mathbf{I} & 0 \\ 0 & \mathbf{I} & 0 & -\mathbf{I} \end{pmatrix} \quad S_2 = \begin{pmatrix} \mathbf{I} & \mathbf{I} & 0 & 0 \\ \mathbf{I} & -\mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & \mathbf{I} \\ 0 & 0 & \mathbf{I} & -\mathbf{I} \end{pmatrix}. \quad (26)$$

If we apply the change of basis corresponding to S_1 followed by the change of basis corresponding to S_2 , then A becomes $S_2^{-1}S_1^{-1}AS_1S_2$ or $(S_1S_2)^{-1}A(S_1S_2)$. Thus, the matrix S_1S_2 plays the same role for two planes of symmetry as S did for one plane of symmetry. As before, this change of basis makes A block diagonal. It is easily verified that

$$S_1S_2 = \begin{pmatrix} \text{I} & \text{I} & \text{I} & \text{I} \\ \text{I} & -\text{I} & \text{I} & -\text{I} \\ \text{I} & \text{I} & -\text{I} & -\text{I} \\ \text{I} & -\text{I} & -\text{I} & \text{I} \end{pmatrix} \quad (27)$$

and that the diagonal submatrices of $(S_1S_2)^{-1}A(S_1S_2)$ are $A_1+A_2+A_3+A_4$, $A_1-A_2+A_3-A_4$, $A_1+A_2-A_3-A_4$, and $A_1-A_2-A_3+A_4$. Thus, two planes of symmetry allow the original system of equations to be replaced by four smaller systems of one-quarter the size, i.e.,

$$\begin{aligned} (A_1 + A_2 + A_3 + A_4) \hat{x}_1 &= \hat{b}_1 \\ (A_1 - A_2 + A_3 - A_4) \hat{x}_2 &= \hat{b}_2 \\ (A_1 + A_2 - A_3 - A_4) \hat{x}_3 &= \hat{b}_3 \\ (A_1 - A_2 - A_3 + A_4) \hat{x}_4 &= \hat{b}_4 \end{aligned} \quad (28a)$$

where

$$\begin{aligned} \hat{b}_1 &= \frac{1}{4}(b_1 + b_2 + b_3 + b_4) \\ \hat{b}_2 &= \frac{1}{4}(b_1 - b_2 + b_3 - b_4) \\ \hat{b}_3 &= \frac{1}{4}(b_1 + b_2 - b_3 - b_4) \\ \hat{b}_4 &= \frac{1}{4}(b_1 - b_2 - b_3 + b_4) \end{aligned} \quad (28b)$$

and

$$\begin{aligned} x_1 &= \hat{x}_1 + \hat{x}_2 + \hat{x}_3 + \hat{x}_4 \\ x_2 &= \hat{x}_1 - \hat{x}_2 + \hat{x}_3 - \hat{x}_4 \\ x_3 &= \hat{x}_1 + \hat{x}_2 - \hat{x}_3 - \hat{x}_4 \\ x_4 &= \hat{x}_1 - \hat{x}_2 - \hat{x}_3 + \hat{x}_4. \end{aligned} \quad (28c)$$

If the right-hand-side also has two planes of symmetry, then it is only necessary to solve the single system

$$(A_1 + A_2 + A_3 + A_4) x_1 = b_1 \quad (29)$$

and set $x_2 = x_3 = x_4 = x_1$.

2.4 Three planes of symmetry

Consider next a problem having three planes of symmetry. We will number the symmetry octants as shown in table 3.

Table 3: Numbering of octants for three planes of symmetry

Octant	x_1	x_2	x_3
1	+	+	+
2	+	+	-
3	+	-	+
4	+	-	-
5	-	+	+
6	-	+	-
7	-	-	+
8	-	-	-

In this case we have three reflection operators

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{pmatrix} \quad (30a)$$

$$\Sigma_2 = \begin{pmatrix} 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{pmatrix} \quad (30b)$$

$$\Sigma_3 = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{pmatrix}. \quad (30c)$$

The three eigenvector matrices are

$$S_1 = \begin{pmatrix} I & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & -I \end{pmatrix} \quad (31)$$

$$S_2 = \begin{pmatrix} I & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & 0 & I & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & -I \end{pmatrix} \quad (32)$$

$$S_3 = \begin{pmatrix} I & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & I \\ 0 & 0 & 0 & 0 & 0 & 0 & I & -I \end{pmatrix}. \quad (33)$$

The product $S_1 S_2 S_3$ is the matrix of basis vectors that block diagonalizes A . It is given by

$$S_1 S_2 S_3 = \begin{pmatrix} I & I & I & I & I & I & I & I \\ I & -I & I & -I & I & -I & I & -I \\ I & I & -I & -I & I & I & -I & -I \\ I & -I & -I & I & I & -I & -I & I \\ I & I & I & I & -I & -I & -I & -I \\ I & -I & I & -I & -I & I & -I & I \\ I & I & -I & -I & -I & -I & I & I \\ I & -I & -I & I & -I & I & I & -I \end{pmatrix}. \quad (34)$$

In this case we can reduce the original system of equations to eight systems of one-eighth

the size, i.e.,

$$\begin{aligned}
(A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8) \hat{x}_1 &= \hat{b}_1 \\
(A_1 - A_2 + A_3 - A_4 + A_5 - A_6 + A_7 - A_8) \hat{x}_2 &= \hat{b}_2 \\
(A_1 + A_2 - A_3 - A_4 + A_5 + A_6 - A_7 - A_8) \hat{x}_3 &= \hat{b}_3 \\
(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 - A_7 + A_8) \hat{x}_4 &= \hat{b}_4 \\
(A_1 + A_2 + A_3 + A_4 - A_5 - A_6 - A_7 - A_8) \hat{x}_5 &= \hat{b}_5 \\
(A_1 - A_2 + A_3 - A_4 - A_5 + A_6 + A_7 - A_8) \hat{x}_6 &= \hat{b}_6 \\
(A_1 + A_2 - A_3 - A_4 - A_5 - A_6 + A_7 + A_8) \hat{x}_7 &= \hat{b}_7 \\
(A_1 - A_2 - A_3 + A_4 - A_5 + A_6 + A_7 - A_8) \hat{x}_8 &= \hat{b}_8
\end{aligned} \tag{35a}$$

where

$$\begin{aligned}
\hat{b}_1 &= \frac{1}{8}(b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8) \\
\hat{b}_2 &= \frac{1}{8}(b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - b_8) \\
\hat{b}_3 &= \frac{1}{8}(b_1 + b_2 - b_3 - b_4 + b_5 + b_6 - b_7 - b_8) \\
\hat{b}_4 &= \frac{1}{8}(b_1 - b_2 - b_3 + b_4 + b_5 - b_6 - b_7 + b_8) \\
\hat{b}_5 &= \frac{1}{8}(b_1 + b_2 + b_3 + b_4 - b_5 - b_6 - b_7 - b_8) \\
\hat{b}_6 &= \frac{1}{8}(b_1 - b_2 + b_3 - b_4 - b_5 + b_6 - b_7 + b_8) \\
\hat{b}_7 &= \frac{1}{8}(b_1 + b_2 - b_3 - b_4 - b_5 - b_6 + b_7 + b_8) \\
\hat{b}_8 &= \frac{1}{8}(b_1 - b_2 - b_3 + b_4 - b_5 + b_6 + b_7 - b_8)
\end{aligned} \tag{35b}$$

and

$$\begin{aligned}
x_1 &= \hat{x}_1 + \hat{x}_2 + \hat{x}_3 + \hat{x}_4 + \hat{x}_5 + \hat{x}_6 + \hat{x}_7 + \hat{x}_8 \\
x_2 &= \hat{x}_1 - \hat{x}_2 + \hat{x}_3 - \hat{x}_4 + \hat{x}_5 - \hat{x}_6 + \hat{x}_7 - \hat{x}_8 \\
x_3 &= \hat{x}_1 + \hat{x}_2 - \hat{x}_3 - \hat{x}_4 + \hat{x}_5 + \hat{x}_6 - \hat{x}_7 - \hat{x}_8 \\
x_4 &= \hat{x}_1 - \hat{x}_2 - \hat{x}_3 + \hat{x}_4 + \hat{x}_5 - \hat{x}_6 - \hat{x}_7 + \hat{x}_8 \\
x_5 &= \hat{x}_1 + \hat{x}_2 + \hat{x}_3 + \hat{x}_4 - \hat{x}_5 - \hat{x}_6 - \hat{x}_7 - \hat{x}_8 \\
x_6 &= \hat{x}_1 - \hat{x}_2 + \hat{x}_3 - \hat{x}_4 - \hat{x}_5 + \hat{x}_6 - \hat{x}_7 + \hat{x}_8 \\
x_7 &= \hat{x}_1 + \hat{x}_2 - \hat{x}_3 - \hat{x}_4 - \hat{x}_5 - \hat{x}_6 + \hat{x}_7 + \hat{x}_8 \\
x_8 &= \hat{x}_1 - \hat{x}_2 - \hat{x}_3 + \hat{x}_4 - \hat{x}_5 + \hat{x}_6 + \hat{x}_7 - \hat{x}_8.
\end{aligned} \tag{35c}$$

If the right-hand-side also has three planes of symmetry, then it is only necessary to solve the single system

$$(A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8) x_1 = b_1 \tag{36}$$

and set $x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = x_1$.

2.5 Finite order rotational symmetry

We will consider next finite order rotational symmetry. For example, a pentagon has rotational symmetry of order five and an octagon has rotational symmetry of order eight. As before we will assume that each symmetry block is numbered in the same order. The rotation operator R has the property

$$Rx = R \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_N \\ x_1 \end{pmatrix} \quad \text{for all } x. \quad (37)$$

Thus, R must have the block form

$$R = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ I & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (38)$$

The eigenvalues of R are the N -th roots of unity. The eigenvector matrix S is given by

$$S = \begin{pmatrix} S_{11} & \cdots & S_{1N} \\ \vdots & & \vdots \\ S_{N1} & \cdots & S_{NN} \end{pmatrix}. \quad (39)$$

where the block S_{mn} is defined by

$$S_{mn} = e^{i\frac{2\pi(m-1)(n-1)}{N}} I. \quad (40)$$

The inverse of S is given by

$$S^{-1} = \frac{1}{N} \begin{pmatrix} S_{11}^* & \cdots & S_{1N}^* \\ \vdots & & \vdots \\ S_{N1}^* & \cdots & S_{NN}^* \end{pmatrix} \quad (41)$$

where $*$ denotes the complex conjugate.

The rotational symmetry implies that A commutes with R . This commutation relation implies that A has the block circulant form

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_N \\ A_N & A_1 & \cdots & A_{N-1} \\ \vdots & & & \vdots \\ A_2 & \cdots & A_N & A_1 \end{pmatrix}. \quad (42)$$

To see this multiply out the blocks of RA and AR as before. The commutation of A with R also implies that $S^{-1}AS$ is block diagonal. The diagonal blocks are given by

$$[S^{-1}AS]_{mm} = \sum_{n=1}^N e^{i\frac{2\pi(m-1)(n-1)}{N}} A_n \quad m = 1, \dots, N. \quad (43)$$

Thus, the original system of equations reduces to the N smaller systems of equations

$$\left(\sum_{n=1}^N e^{i\frac{2\pi(m-1)(n-1)}{N}} A_n \right) \hat{x}_m = \hat{b}_m \quad m = 1, \dots, N \quad (44a)$$

where

$$\hat{b}_m = \frac{1}{N} \sum_{n=1}^N e^{-i\frac{2\pi(m-1)(n-1)}{N}} b_n \quad (44b)$$

and

$$x_m = \sum_{n=1}^N e^{i\frac{2\pi(m-1)(n-1)}{N}} \hat{x}_n \quad m = 1, \dots, N. \quad (44c)$$

If the right-hand-side vector b also has N -th order rotational symmetry, then it is only necessary to solve the single system

$$(A_1 + A_2 + \dots + A_N)x_1 = b_1 \quad (44d)$$

and set $x_2 = x_3 = \dots = x_N = x_1$.

For those familiar with Fourier analysis it is easily seen that the symmetry reductions in this section could also be obtained by using discrete Fourier transforms.

2.6 Finite order rotational symmetry with one additional plane of symmetry

Often there is an additional plane of symmetry in problems having finite order rotational symmetry. This case can be handled by successively applying the results of finite order rotational symmetry and those for one plane of symmetry. Assume that each of the rotational blocks A_m of A is further partitioned into two parts A_m^+ and A_m^- consistent with the additional plane of symmetry. Likewise, the blocks of b are partitioned into b_m^+ , b_m^- and the blocks of x_m are partitioned into x_m^+ , x_m^- . Then the original system of equations can be reduced to the $2N$ smaller systems

$$\left(\sum_{n=1}^N e^{i\frac{2\pi(m-1)(n-1)}{N}} (A_n^+ + A_n^-) \right) \hat{x}_m^+ = \hat{b}_m^+ \quad m = 1, \dots, N \quad (45a)$$

$$\left(\sum_{n=1}^N e^{i\frac{2\pi(m-1)(n-1)}{N}} (A_n^+ - A_n^-) \right) \hat{x}_m^- = \hat{b}_m^- \quad m = 1, \dots, N \quad (45b)$$

where

$$\hat{b}_m^+ = \frac{1}{2N} \sum_{n=1}^N e^{-i\frac{2\pi(m-1)(n-1)}{N}} (b_n^+ + b_n^-) \quad (45c)$$

$$\hat{b}_m^- = \frac{1}{2N} \sum_{n=1}^N e^{-i\frac{2\pi(m-1)(n-1)}{N}} (b_n^+ - b_n^-) \quad (45d)$$

and

$$x_m^+ = \sum_{n=1}^N e^{i\frac{2\pi(m-1)(n-1)}{N}} (\hat{x}_m^+ + \hat{x}_m^-) \quad (45e)$$

$$x_m^- = \sum_{n=1}^N e^{i\frac{2\pi(m-1)(n-1)}{N}} (\hat{x}_m^+ - \hat{x}_m^-). \quad (45f)$$

If the right-hand-side also has the same symmetry, then it is only necessary to solve the single system

$$\left(\sum_{n=1}^N (A_n^+ + A_n^-) \right) x_1^+ = b_1^+ \quad (46)$$

and equate the other blocks x_m^+ , x_m^- to x_1^+ .

2.7 Eigenproblems

The symmetry structure of A can also be used to reduce the computational effort in eigenproblems. Suppose A can be reduced to block diagonal form by $S^{-1}AS$, i.e.,

$$S^{-1}AS = \text{diag}(\hat{A}_1, \dots, \hat{A}_N). \quad (47)$$

Let E_n be the eigenmatrix corresponding to \hat{A}_n , and let Λ_n be the corresponding diagonal matrix of eigenvalues. Then

$$\hat{A}_n E_n = E_n \Lambda_n \quad \text{for all } n. \quad (48)$$

These eigen relations can be combined into the following matrix relation

$$\text{diag}(\hat{A}_1, \dots, \hat{A}_N) \cdot \text{diag}(E_1, \dots, E_N) = \text{diag}(E_1, \dots, E_N) \cdot \text{diag}(\Lambda_1, \dots, \Lambda_N). \quad (49)$$

In view of equation (47), this relation can be written

$$S^{-1}AS \cdot \text{diag}(E_1, \dots, E_N) = \text{diag}(E_1, \dots, E_N) \cdot \text{diag}(\Lambda_1, \dots, \Lambda_N). \quad (50)$$

or

$$AS \cdot \text{diag}(E_1, \dots, E_N) = S \cdot \text{diag}(E_1, \dots, E_N) \cdot \text{diag}(\Lambda_1, \dots, \Lambda_N). \quad (51)$$

Thus, $S \cdot \text{diag}(E_1, \dots, E_N)$ is the eigenmatrix of A corresponding to the eigenvalues along the diagonal of $\text{diag}(\Lambda_1, \dots, \Lambda_N)$.