Notes on Harmonic Analysis

Dr. George W Benthien

May 22, 2006

E-mail: george@gbenthien.net

Contents

2

1 Introduction

Harmonic analysis is concerned with the representation of functions by trigonometric sums or integrals. These trigonometric representations are usually referred to as Fourier series or Fourier integrals. Harmonic expansions have proven to be useful in such diverse fields as astronomy, acoustics, optics, signal processing, image processing, and data compression. In this paper we will discuss some of the practical aspects involved in using these expansions. The material in these notes is based largely on the work of Cornelius Lanczos [7, 8]. The discussion of Fast Fourier Transforms draws heavily on the material in the book Numerical Recipes [11].

1.1 Historical Background

Trigonometric expansions have a long history. The early work in the eighteenth century was focused primarily on the solution of two problems. The first was the vibration of a taut string anchored at both ends. The second was the interpolation of planetary orbits between observation points.

D'Alembert (1747) derived the following wave equation for the vibrating string problem

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1.1}$$

where u is the displacement of the string and c is a constant related to the tension of the string. He showed that a general solution to this wave equation is given by

$$u(x,t) = f(ct + x) + g(ct - x) \tag{1.2}$$

where f and g are arbitrary twice differentiable functions. Since the displacement at the end $x = 0$ is zero, it follows that $g = -f$, i.e.,

$$u(x,t) = f(ct + x) - f(ct - x). \tag{1.3}$$

Since the displacement af the other end $x = L$ is also zero, it follows that

$$f(ct + L) = f(ct - L) \tag{1.4}$$

and hence that f is periodic with period $2L$. He gave several examples of periodic functions that could be used for f including the trigonometric functions.

In 1749 Euler proposed using the function $f(x) = \sin \frac{n\pi x}{L}$ in D'Alembert's solution, yielding

$$u(x,t) = f(ct+x) - f(ct-x) = 2\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}. \qquad (1.5)$$

He also pointed out that any linear combination of solutions of this form is also a solution.

Daniel Bernoulli (1753) suggested that the infinite series expansion

$$u(x,t) = 2\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \qquad (1.6)$$

is a general solution of D'Alembert's wave equation with fixed end conditions, and hence that any initial shape function could be expanded in terms of an infinite series of sine functions. Lagrange (1759) discretized the vibrating string and obtained a discrete sine series as a solution.

In 1777, Euler, working on a problem in astronomy, obtained the coefficients of a cosine series using orthogonality. That is, he used the orthogonality relations

$$\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{for } m \neq n \\ L/2 & \text{for } m = n \neq 0 \\ L & \text{for } m = n = 0 \end{cases}$$

to show that the coefficients in the series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

Euler, Lagrange, Clairaut, and others used trigonometric sums to interpolate planetary orbits between a given set of observation points. Problems of this type lead naturally to periodic functions. Since there are only a finite number of observations, the computation of the coefficients in the trigonometric interpolation expansion led to what we now call the Discrete Fourier Transform. Clairaut (1754) is credited with providing the first explicit formula for the DFT. He considered functions with even symmetry which lead to discrete cosine expansions. As we saw previously, Lagrange (1759) employed discrete sine expansions. Gauss (1805) wrote a paper on trigonometric interpolation that handled the general case and also contained a fast algorithm for computing the DFT [6]. His algorithm is equivalent to what is now called the Fast Fourier Transform (FFT). A number of authors have rediscovered the FFT since the time of Gauss. In particular, Lanczos and

Danielson (1942) published a nice FFT algorithm for 2^n samples that will described in a later section [4]. In 1965 Cooley and Tukey published an FFT algorithm for any composite integer number of samples [3]. Although a number of mathematicians discovered the FFT prior to the paper by Cooley and Tukey, it was certainly this paper that triggered the widespread usage of the FFT.

Trigonometric series are now named for Fourier, but we have seen that a number of mathematicians used series of this type prior to Fourier. Fourier did, however, make a number of important contributions to harmonic analysis. Fourier (1822) looked at the problem of describing the evolution of the temperature $T(x,t)$ in a thin wire of length π stretched between $x = 0$ and $x = \pi$ when the ends are held at zero temperature. He proposed that the initial temperature $T(x,0)$ could be expanded in a series of sine functions

$$T(x,0) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.7}$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} T(x,0) \sin nx \, dx. \tag{1.8}$$

He then showed that the solution to the heat equation with these boundary and initial conditions is given by

$$T(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx. \tag{1.9}$$

In deriving this expression, Fourier used D. Bernoulli's method of separation of variables, a technique widely used today. Fourier recognized that the initial temperature distribution could contain jump discontinuities unlike the initial shape function in the string problem. He suggested that even discontinuous functions could be expanded in trigonometric series. He also recognized that an infinite series expansion could represent a function in an interval and disagree with it outside that interval. Although Fourier was not the first to employ Fourier series expansions, he was the first to define and use the Fourier integral.

As we have seen, a number of mathematicians (Euler, Lagrange, Bernoulli, Clairaut, Fourier, etc.) employed trigonometric expansions to solve various problems. However, the arguments used to support these expansions were more intuitive than rigorous. Dirichlet (1829) examined more carefully the convergence of Fourier series expansions. He showed that the $N - th$ partial sum of a trigonometric expansion could be written as

$$S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N + 1/2)t}{\sin t/2} f(x - t) \, dt. \tag{1.10}$$

Using this he was able to show that a function defined on an interval could be expanded in a Fourier series if it was piecewise continuous and had bounded variation. A function f defined on an interval $[a, b]$ is said to have *bounded variation* if there exists a constant B such that for any partition $a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a, b]$ we have

$$\sum_{i=0}^{N} |f(x_i) - f(x_{i-1})| \leq B.$$

Jordan showed that a function has bounded variation if and only if it can be expressed as the difference of two non-decreasing functions.

Fejér (1904) applied a different method for summing infinite series that greatly extended the validity of harmonic expansions. He showed that it is only necessary to require that f is absolutely integrable on $[-\pi, \pi]$ if summation of the series is defined differently. He called an infinite series $\sum_{k=1}^{\infty} a_k$ summable if the sequence

$$S_1, \ \frac{S_1 + S_2}{2}, \ \frac{S_1 + S_2 + S_3}{3}, \ \ldots$$

converges. Here S_n is the n-th partial sum $S_n = \sum_{k=1}^{n} a_k$. It can be shown that a series that converges in the conventional sense is also summable with the same limit. There are, however, divergent series that are summable. For example, the series expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

is divergent at $x = 1$ where the right-hand-side becomes $1 - 1 + 1 - 1 + \ldots$. However, it is summable to the correct value $1/2$.

As we have seen, harmonic analysis has a long history and has occupied the attention of many famous mathematicians. This topic has also generated a great deal of controversy. Much of the controversy centered around the question: What constitutes a function? Initially the concept of a function was more like what we would call an expression. It included algebraic expressions, expressions involving known functions such as sines, cosines and logarithms, and eventually included some power series. Functions were often closely associated with physical quantities. For example, D'Alembert, in his study of the wave equation, would only consider functions that were differentiable. Euler extended the function concept to include functions that could be graphed with a pencil (without lifting the pencil). These functions were continuous, but could have discontinuous slopes and could be defined by different expressions in different intervals. Fourier extended the function concept to include functions with step discontinuities. Dirichlet, in addition to analysing the convergence of Fourier series, also gave a definition of a function that is close to what we use today. Thus, the concept of a function as well as other concepts like continuity and convergence, grew out of the study of trigonometric expansions. Let us now look at some of the properties of harmonic expansions.

2 Fourier Series

Let f be a real-valued function defined on the interval $[-\pi, \pi]$. Then the Fourier series expansion of f is given by

$$
\begin{aligned}
f(x) = \tfrac{1}{2}a_0 &+ a_1 \cos x + a_2 \cos 2x + \ldots \\
&+ b_1 \sin x + b_2 \sin 2x + \ldots
\end{aligned} \tag{2.1}
$$

where

$$
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \tag{2.2}
$$

$$
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \tag{2.3}
$$

The functions $\sin nx$ and $\cos nx$ are orthogonal in the sense that

$$
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \qquad m \neq n
$$

$$
\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \qquad \text{for all } m, n
$$

$$
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \qquad m \neq n.
$$

At a point x where there is a jump discontinuity, the series in equation (2.1) converges to $[(f(x + 0) + f(x - 0)]/2$. Here $F(x + 0)$ and $f(x - 0)$ are the limits of $f(u)$ as u approaches x from above and below respectively. I still find it remarkable that such a wide variety of functions can be represented on an interval by Fourier series.

2.1 Exponential Form of Fourier Series

The Fourier expansion of f can also be written in the form

$$
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{2.4}
$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx. \tag{2.5}$$

The coefficients c_n are generally complex. The functions e^{inx} are orthogonal in the sense that

$$\int_{-\pi}^{\pi} e^{imx}e^{-inx}\,dx = 0 \qquad m \neq n.$$

2.2 Change of Interval

In these notes we will generally restrict ourselves to functions defined on the interval $[-\pi, \pi]$. Functions defined on other finite intervals can be reduced to this case by a linear change of variable. For example, suppose a function $f(x)$ is defined on the interval $[a, b]$. Consider the change of variable

$$x = \frac{b-a}{2\pi}y + \frac{a+b}{2} \qquad \text{or} \qquad y = \frac{2\pi}{b-a}\left(x - \frac{a+b}{2}\right) \tag{2.6}$$

where y lies in the interval $[-\pi, \pi]$. Let $\hat{f}$ be the function defined on $[-\pi, \pi]$ by

$$\hat{f}(y) = f(x) = f\left(\frac{b-a}{2\pi}y + \frac{a+b}{2}\right). \tag{2.7}$$

The Fourier expansion of $\hat{f}$ can be written

$$\hat{f}(y) = \sum_{n=-\infty}^{\infty} c_n e^{iny} \tag{2.8}$$

where c_n is given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(y)e^{-iny}\,dy. \tag{2.9}$$

Making the change of variables (2.6) in equations (2.8) and (2.9), we get

$$f(x) = \hat{f}(y) = \sum_{n=-\infty}^{\infty} c_n e^{iny} = \sum_{n=-\infty}^{\infty} c_n e^{-i\pi n(a+b)/(b-a)} e^{i2\pi nx/(b-a)} \tag{2.10a}$$

$$c_n e^{-i\pi n(a+b)/(b-a)} = \frac{1}{b-a} \int_a^b f(x)e^{-i2\pi nx/(b-a)}\,dx. \tag{2.10b}$$

If we define

$$\hat{c}_n = c_n e^{-i\pi n(a+b)/(b-a)}, \tag{2.11}$$

then we have

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{c}_n e^{i2\pi nx/(b-a)} \tag{2.12a}$$

$$\hat{c}_n = \frac{1}{b-a} \int_a^b f(x) e^{-i2\pi nx/(b-a)} \, dx. \tag{2.12b}$$

Equations (2.12a) and (2.12b) define the Fourier series expansion of f on the interval $[a, b]$.

2.3 Even and Odd Functions

The function f can be expressed as

$$f(x) = g(x) + h(x) \tag{2.13}$$

where

$$g(x) = \tfrac{1}{2}[f(x) + f(-x)] \tag{2.14}$$
$$h(x) = \tfrac{1}{2}[f(x) - f(-x)]. \tag{2.15}$$

The function g is an *even function* in the sense that $g(-x) = g(x)$. The function h is an *odd function* in the sense that $h(-x) = -h(x)$. Moreover, if f has a Fourier expansion like that in equation (2.1), then

$$g(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots \tag{2.16}$$
$$h(x) = b_1 \sin x + b_2 \sin 2x + \ldots \tag{2.17}$$

and

$$a_n = \frac{2}{\pi} \int_0^\pi g(x) \cos nx \, dx \tag{2.18}$$

$$b_n = \frac{2}{\pi} \int_0^\pi h(x) \sin nx \, dx. \tag{2.19}$$

If the function f is defined on the interval $[0.\pi]$, then it can be extended to the interval $[-\pi, \pi]$ as either an odd function or as an even function. Thus, f can be represented on $[0, \pi]$ in terms of a cosine series or a sine series.

9

2.4 Rate of Convergence

In numerical computations we are not only interested in the convergence of an infinite series, but also in the rate at which it converges. Suppose f is an infinitely differentiable function on the interval $[-\pi, \pi]$ that can be expanded in a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{2.20}$$

with coefficients c_n given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx. \tag{2.21}$$

Integrating the right-hand-side of equation (2.21) by parts three times, we obtain

$$2\pi c_n = \frac{i(-1)^n}{n} [f(\pi) - f(-\pi)] + \frac{(-1)^n}{n^2} [f'(\pi) - f'(-\pi)] -$$
$$\frac{i(-1)^n}{n^3} [f''(\pi) - f''(-\pi)] + \frac{i}{n^3} \int_{-\pi}^{\pi} f'''(x) e^{-inx} \, dx. \tag{2.22}$$

We can learn a lot about the rate of convergence of the Fourier series from equation (2.22). If $f(-\pi) \neq f(\pi)$, then the periodic extension of f will have jump discontinuities at the end points of the interval and the convergence rate will be $1/n$. However, if $f(-\pi) = f(\pi)$, the extension will be continuous and the convergence rate will be at least $1/n^2$. If, in addition, $f'(-\pi) = f'(\pi)$, then the periodic extension will have a continuous derivative and the convergence rate will be at least $1/n^3$. Thus the convergence rate depends strongly on the smoothness of the periodic extension of f. Knowing this, it is often possible to increase the convergence rate by modifying the function in such a way as to increase the smoothness of the periodic extension. Several examples of this will be shown later. A particularly important example is shown in subsection 4.4.1 dealing with the calculation of impulse responses.

Let us now look at the rate of convergence of Fourier Sine and Cosine expansions. Suppose f is defined on $[0, \pi]$. If we extend f to $[-\pi, \pi]$ as an even function, then f can be expanded in a cosine series, i.e.,

$$f(x) = \tfrac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \dots. \tag{2.23}$$

The coefficients a_n are given by

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx. \tag{2.24}$$

Integrating equation (2.24) by parts twice, we obtain

$$\frac{\pi}{2} a_n = \frac{(-1)^n f'(\pi) - f'(0)}{n^2} - \frac{1}{n^2} \int_0^\pi f''(x) \cos nx \, dx. \tag{2.25}$$

Unless f satisfies certain end conditions on $[0, \pi]$, a cosine expansion of f will converge like $1/n^2$.

Suppose f is extended as an odd function. Then f can be expanded in a sine series, i.e.,

$$f(x) = b_1 \sin x + b_2 \sin 2x + \dots. \tag{2.26}$$

The coefficients b_n are given by

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx. \tag{2.27}$$

Integrating equation (2.27) by parts three times, we obtain

$$\frac{\pi}{2} b_n = \frac{f(0) - (-1)^n f(\pi)}{n} + \frac{(-1)^n f''(\pi) - f''(0)}{n^3} - \frac{1}{n^3} \int_0^\pi f'''(x) \cos nx \, dx. \tag{2.28}$$

Unless f satisfies certain end conditions on $[0, \pi]$, a sine expansion of f will converge like $1/n$. However, if $f(0) = f(\pi) = 0$, then a sine expansion converges like $1/n^3$. In general f will not vanish at 0 and π. However, we can easily modify f so that it does. The linear function u defined by

$$u(x) = f(0) + \frac{f(\pi) - f(0)}{\pi} x \tag{2.29}$$

has the property that $u(0) = f(0)$ and $u(\pi) = f(\pi)$. Therefore, the function $f - u$ vanishes at 0 and π and hence can be expanded in a sine series with convergence rate $1/n^3$.

2.5 Differentiation of Fourier Series and Sigma Factors

Suppose f has the Fourier expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{2.30}$$

Let f_m be the partial sum defined by

$$f_m(x) = \sum_{n=-(m-1)}^{m-1} c_n e^{inx}. \tag{2.31}$$

The residual η_m is defined by

$$\eta_m(x) = \sum_{n=m}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx}). \tag{2.32}$$

Equation (2.32) can be rewritten as follows:

$$\eta_m = e^{imx} \sum_{n=0}^{\infty} c_{m+n} e^{inx} + e^{-imx} \sum_{n=0}^{\infty} c_{-m-n} e^{-inx}. \tag{2.33}$$

Define ϕ_m and ψ_m by

$$\phi_m(x) = \sum_{n=0}^{\infty} c_{m+n} e^{inx} \tag{2.34}$$

$$\psi_m(x) = \sum_{n=0}^{\infty} c_{-m-n} e^{-inx}. \tag{2.35}$$

Then Equation (2.33) can be written

$$\eta_m(x) = e^{imx} \phi_m(x) + e^{-imx} \psi_m(x). \tag{2.36}$$

The functions ϕ_m and ψ_m are generally smooth functions which do not show any rapid oscillations. On the other hand, e^{imx} and e^{-imx} are rapidly oscillating functions when m is large. Thus, the error behaves like a modulated carrier wave of high frequency. If we formally differentiate f_m and compare it with $f'(x)$, we obtain the error

$$\eta'_m(x) = im e^{imx}\phi_m(x) + e^{imx}\phi'_m(x) - im e^{-imx}\psi_m(x) + e^{-imx}\psi'_m(x). \qquad (2.37)$$

The primed terms arise from differentiation of the modulation and do not cause any serious difficulty. However, the other terms contain the factor m that comes from differentiating the carrier. For large m these terms can be significant. In fact they can cause the differentiated series to diverge.

Suppose we replace the process of differentiation by the following central difference process:

$$D_m f(x) = \frac{f(x + \pi/m) - f(x - \pi/m)}{2\pi/m}. \qquad (2.38)$$

For large m, D_m is a good approximation to the derivative. Since

$$e^{\pm im(x \pm \pi/m)} = -e^{\pm imx}, \qquad (2.39)$$

it follows that

$$D_m \eta_m(x) = -e^{imx} D_m \phi_m(x) - e^{-imx} D_m \psi_m(x). \qquad (2.40)$$

Notice that there are no factors of m in this expression since the operator D_m picks out two points on the carrier wave that are in phase ($\pm 180°$ away from the phase at x). Moreover,

$$D_m e^{inx} = in e^{inx} \frac{\sin(n\pi/m)}{n\pi/m} \qquad (2.41)$$

$$D_m e^{-inx} = -in e^{-inx} \frac{\sin(n\pi/m)}{n\pi/m}. \qquad (2.42)$$

Therefore, to apply D_m to the sum defining f_m in equation (2.31) we can differentiate the sum formally term by term and then apply the factor

$$\sigma_n = \frac{\sin(n\pi/m)}{n\pi/m} \qquad (2.43)$$

to the terms corresponding to $\pm n$. The factor σ_n is 1 for $n = 0$ and then decreases monotonically with increasing n. It is almost zero at the highest subscript $m - 1$. Figure 2.1 is a plot of σ_n vs. n for $m = 30$.

This attenuation of the higher harmonics counteracts the tendency for the series to become divergent. Since any function can be considered as the derivative of its integral, sigma factors can be applied to any Fourier expansion in order to increase its convergence rate. We will look at the effect of applying these sigma factors in the following examples.
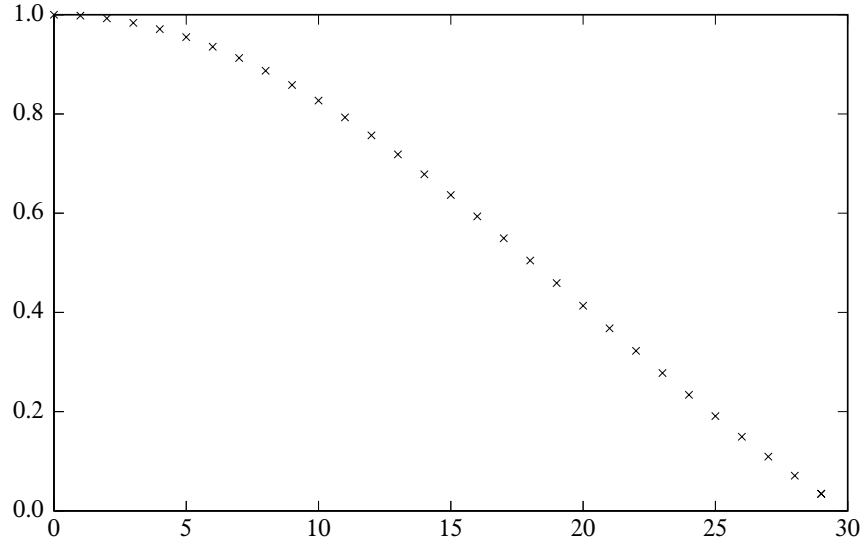
13

Figure 2.1: Plot of σ_n vs. n for $m = 30$.

Example 1. The formal fourier series of a delta function is given by

$$\delta(x) = \frac{1}{\pi}\left(\tfrac{1}{2} + \cos x + \cos 2x + \dots\right). \tag{2.44}$$

This series does not converge at any point. If sigma factors are applied to the first m terms of this series, we get

$$\delta_m(x) = \frac{1}{\pi}\left(\tfrac{1}{2} + \sigma_1 \cos x + \sigma_2 \cos 2x + \cdots + \sigma_{m-1} \cos(m-1)x\right). \tag{2.45}$$

A plot of this function for $m = 30$ is shown in figure 2.2.

In addition to converting divergent Fourier series into convergent ones, sigma factors can also be used to increase the convergence rate of a slowly convergent Fourier series.

Example 2. Consider the square wave defined by

$$\begin{aligned} f(-x) &= -f(x) \\ f(x) &= \tfrac{1}{2} \quad (0 < x < \pi) \\ f(0) &= f(\pi) = 0. \end{aligned} \tag{2.46}$$

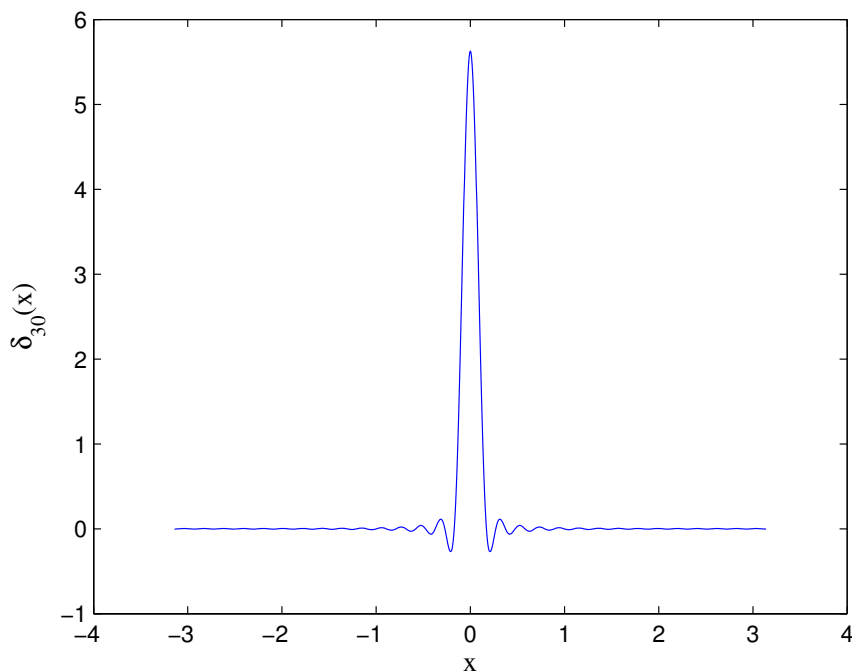This function has the slowly convergent Fourier series expansion

14

Figure 2.2: Approximation of delta function using sigma factors.

$$f(x) = \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right). \tag{2.47}$$

Figure 2.3 shows the approximation to the square wave using 30 terms in the above series.

Notice the Gibbs phenomena near the discontinuities where the approximation overshoots the square wave and then rings for a while. Figure 2.4 shows the approximation obtained using Fejér's arithmetic mean summation method.

This method removes the Gibbs phenomena, but is slowly convergent near the discontinuities. Figure 2.5 shows the effect of applying sigma factors to the approximation.

This cuts down the Gibbs phenomena considerably and has better convergence near the discontinuities than Fejér's method.

Sigma factors can sometimes be used to obtain trigonometric series approximations to functions that are not integrable. For example, the log function $\ln x$ is integrable and has a Fourier series approximation. The function $1/x$ is not integrable, but is the derivative of $\ln x$. Thus, we can differentiate the expansion of $\ln x$ and apply sigma factors to obtain an approximation to $1/x$.

The sigma factor method can be looked upon in a different way. Instead of replacing the differentiation operator $\frac{d}{dx}$ by the difference operator D_m, we could instead replace the function f by a locally smoothed function $\bar{f}$ defined by
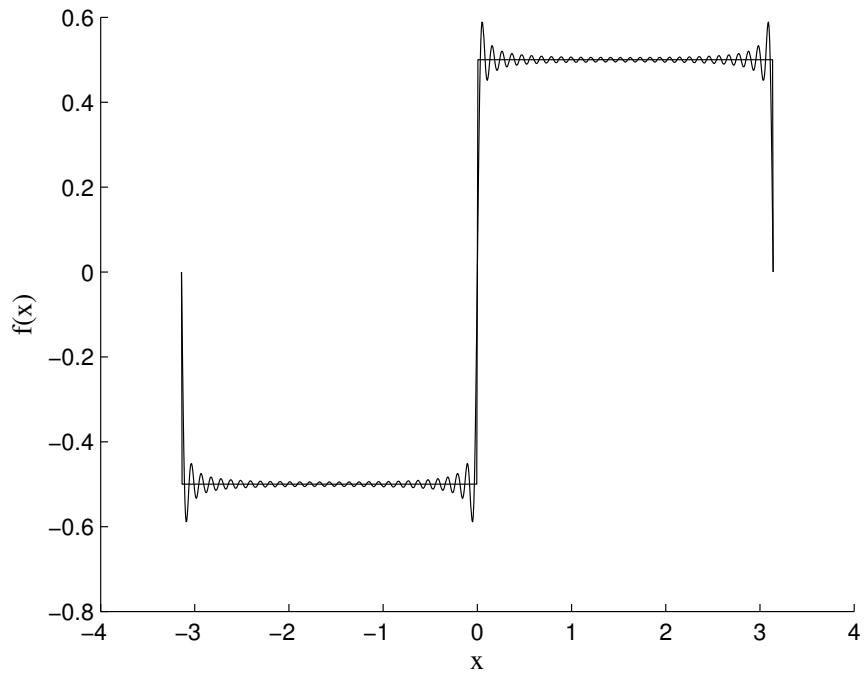
15

Figure 2.3: Fourier sine series approximation of square wave.
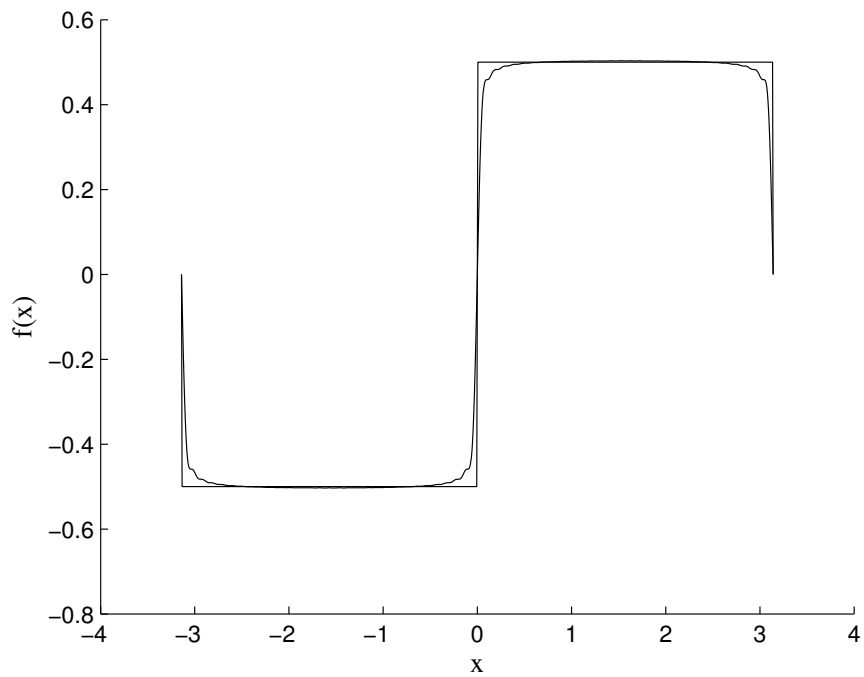
Figure 2.4: Approximation of square wave using Fejér's arithmetic mean method
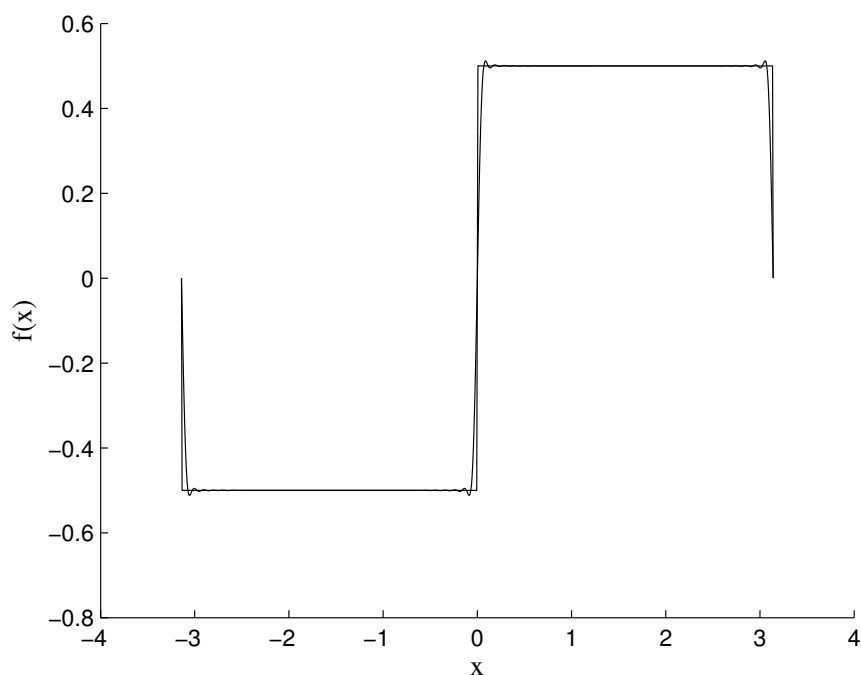
Figure 2.5: Approximation of square wave using sigma factors

$$\bar{f}(x) = \frac{m}{2\pi} \int_{-\pi/m}^{\pi/m} f(x+t) \, dt. \tag{2.48}$$

Differentiating (2.48), we get

$$\frac{d}{dx}\bar{f}(x) = \frac{m}{2\pi} \int_{-\pi/m}^{\pi/m} \frac{d}{dx} f(x+t) \, dt = \frac{m}{2\pi} \int_{-\pi/m}^{\pi/m} f'(x+t) \, dt$$

$$= \frac{m}{2\pi} \int_{x-\pi/m}^{x+\pi/m} f'(t) \, dt = \frac{f(x+\pi/m) - f(x-\pi/m)}{\pi/m} = D_m f(x). \tag{2.49}$$

Notice that as m increases, the interval over which f is averaged gets smaller and smaller.

Extension of the Concept of Convergence   Usually when we talk about the sum of an infinite series such as a Fourier series we are thinking of a fixed sequence of terms $a_1, a_2, \ldots$, and we define the sum to be the limit of the partial sums S_n as $n \to \infty$. With the method of sigma factors the weighted terms actually change as the number of terms in the partial sums increases. Thus, the single sequence $a_1, a_2, \ldots$ is replaced by the triangular array

$$
\begin{array}{llll}
a_{11} & & & \\
a_{21} & a_{22} & & \\
a_{31} & a_{32} & a_{33} & \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

of terms. The partial sums S_n are now defined by

$$
S_n = a_{n1} + a_{n2} \cdots + a_{nn},
$$

and the sum S is again defined as $S = \lim_{n \to \infty} S_n$. This flexibility in adjusting the terms as their number increases can often lead to better approximations for a given number of terms.

2.6 The Trapezoidal Rule and Fourier Series

One of the simplest methods for estimating the area under a curve is to connect equally spaced ordinates by straight lines and to approximate the area under the curve by the area under the polygonal line (see figure 2.6).
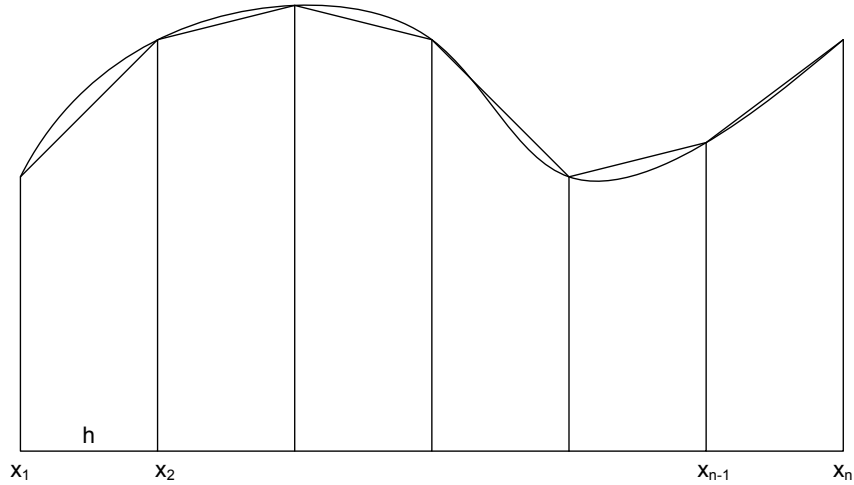
Figure 2.6: Approximating the area under a curve using the trapezoidal rule.

This method is known as the trapezoidal rule since the area under the polygonal line is the sum of areas of trapezoids, i.e.,

$$
\int_{x_1}^{x_n} f(x)\, dx \doteq h \left[\frac{f(x_1) + f(x_2)}{2} + \frac{f(x_2) + f(x_3)}{2} + \cdots \frac{f(x_{n-1}) + f(x_n)}{2} \right]
$$
$$
= h \left[\tfrac{1}{2} f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2} f(x_n) \right].
\tag{2.50}
$$

In this section we will show the close connection between the trapezoidal rule and trigonometric expansions. We will use this connection to develop a modified trapezoidal rule with greater accuracy.

Suppose f is defined on the interval $[0, 1]$. Let us expand f into a Fourier cosine series, i.e.,

$$
f(x) = \tfrac{1}{2} a_0 + a_1 \cos \pi x + a_2 \cos 2\pi x + \dots .
\tag{2.51}
$$

Using the trapezoidal rule we approximate the area A under the curve by the area $\bar{A}$ under the inscribed polygon, i.e.,

$$
\bar{A} = \frac{1}{n} \left[\tfrac{1}{2} f(0) + f\left(\frac{1}{n} \right) + \cdots + f\left(\frac{n-1}{n} \right) + \tfrac{1}{2} f(1) \right].
\tag{2.52}
$$

Substituting (2.51) into (2.52) and rearranging terms, we see that the coefficient of a_0 is one half, the coefficient of the odd term a_{2k-1} is

$$\frac{1}{n} \sum_{m=1}^{n-1} \cos\left(\frac{(2k-1)m\pi}{n}\right),$$

and the coefficient of the even term a_{2k} is

$$\frac{1}{n} \sum_{m=0}^{n-1} \cos\left(\frac{2km\pi}{n}\right).$$

It can be shown that

$$\frac{1}{n} \sum_{m=1}^{n-1} \cos\left(\frac{(2k-1)m\pi}{n}\right) = 0 \tag{2.53}$$

and

$$\frac{1}{n} \sum_{m=0}^{n-1} \cos\left(\frac{2km\pi}{n}\right) = \begin{cases} 0 & \text{for } k \text{ not a multiple of } n \\ 1 & \text{for } k \text{ a multiple of } n. \end{cases} \tag{2.54}$$

Therefore,

$$\bar{A} = \tfrac{1}{2}a_0 + a_{2n} + a_{4n} + \dots. \tag{2.55}$$

The Fourier coefficient a_0 is given by

$$a_0 = 2 \int_0^1 f(x)\, dx. \tag{2.56}$$

Therefore,

$$\bar{A} = \int_0^1 f(x)\, dx + a_{2n} + a_{4n} + \dots. \tag{2.57}$$

It can be seen from equation (2.57) that the error in the trapezoidal rule depends on how fast the Fourier cosine series converges. For many oscillating integrands the trapezoidal rule gives surprisingly good results. Consider the function shown in figure 2.7.

Looking at the curve you would think that it would require at least 20 points to get an accurate value for the integral using the trapezoidal rule. However, since the curve was generated by the formula
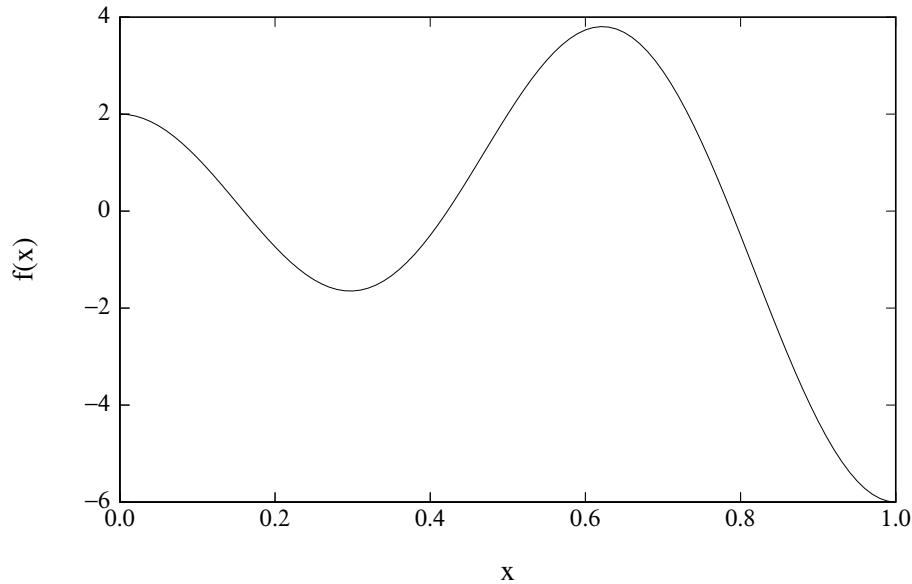
Figure 2.7: Example function for trapezoidal integration.

$$f(x) = \cos(\pi x) - 2\cos(2\pi x) + 3\cos(3\pi x), \tag{2.58}$$

it only requires four points to get an exact value. We will now develop a more accurate trapezoidal rule.

Integrating the expression for a_{2n} by parts, we get

$$\tfrac{1}{2}a_{2n} = \int_0^1 f(x)\cos(2n\pi x)\,dx = \frac{f'(1) - f'(0)}{4n^2} - \frac{1}{4n^2}\int_0^1 f''(x)\cos(2n\pi x)\,dx. \tag{2.59}$$

Thus, a_{2n} converges like $1/n^2$ unless $f'(1) = f'(0)$. In general, f doesn't have this property. However, the modified function g defined by

$$g(x) = f(x) - \frac{f'(1) - f'(0)}{2}x^2 \tag{2.60}$$

does have the property $g'(1) = g'(0)$. Therefore, we can use the trapezoidal rule on g and obtain an approximation to the integral of f using

21

$$\int_0^1 f(x)\,dx \doteq \frac{1}{n}\left[\tfrac{1}{2}g(0) + g\left(\frac{1}{n}\right) + \cdots + g\left(\frac{n-1}{n}\right) + \tfrac{1}{2}g(1)\right] + \frac{f'(1) - f'(0)}{2}\int_0^1 x^2\,dx$$

$$= \frac{1}{n}\left[\tfrac{1}{2}g(0) + g\left(\frac{1}{n}\right) + \cdots + g\left(\frac{n-1}{n}\right) + \tfrac{1}{2}g(1)\right] + \frac{f'(1) - f'(0)}{6}. \qquad (2.61)$$

If we substitute the definition of g from (2.60) into (2.61), we get

$$\int_0^1 f(x)\,dx \doteq \frac{1}{n}\left[\tfrac{1}{2}f(0) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + \tfrac{1}{2}f(1)\right] -$$
$$\frac{f'(1) - f'(0)}{2n^3}\left[\sum_{k=1}^n k^2 - \frac{n^2}{2}\right] + \frac{f'(1) - f'(0)}{6}. \qquad (2.62)$$

Since

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad (2.63)$$

equation (2.62) becomes

$$\int_0^1 f(x)\,dx \doteq \frac{1}{n}\left[\tfrac{1}{2}f(0) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + \tfrac{1}{2}f(1)\right] - \frac{f'(1) - f'(0)}{12n^2}. \qquad (2.64)$$

By making a linear change of variable in the integral, we can show that

$$\int_a^b f(x)\,dx \doteq \frac{b-a}{n}\left[\tfrac{1}{2}f(a) + f\left(a + \frac{b-a}{n}\right) + \cdots + f\left(a + (n-1)\frac{b-a}{n}\right) + \tfrac{1}{2}f(b)\right] -$$
$$\frac{f'(b) - f'(a)}{12}\left(\frac{b-a}{n}\right)^2. \qquad (2.65)$$

This relation is known as the trapezoidal rule with end correction. It usually gives much greater accuracy than the trapezoidal rule if the values of the derivative f' at the end points are known or can be accurately approximated.

Example 3. Suppose we want to approximate the following integral

$$\int_0^4 e^x \, dx.$$

The following table gives the function values of e^x at increments of 0.5:

Table 2.1: Function values of e^x

x	e^x
0.0	1.000000
0.5	1.648721
1.0	2.718281
1.5	4.481689
2.0	7.389056
2.5	12.182494
3.0	20.085537
3.5	33.115452
4.0	54.598150

The exact answer for the integral is given by

$$\int_0^4 e^x \, dx = e^4 - 1 \doteq 53.59815$$

The trapezoidal rule gives the approximate answer 54.71015. This corresponds to a relative error of 2.075%. The trapezoidal rule with end corrections gives the approximate answer 53.59352. This corresponds to a relative error of -0.0086%.

3 Chebyshev Polynomials

To expand a function defined on a finite interval in a Fourier series it is necessary to extend this function in some way in order to make it periodic. The convergence of the series is governed by how smooth we can make this extension. In practice it is usually difficult to get more smoothness than continuity of the first derivative. In this section we will discuss a modified form of the Fourier series that gets much better convergence rates.

3.1 Basic Properties

Suppose f is an infinitely smooth function on the interval $[-1, 1]$, but has no special boundary conditions. Let us make the change of variable

$$x = \cos\theta \qquad 0 \le \theta \le \pi \tag{3.1}$$

and define

$$\phi(\theta) = f(\cos\theta). \tag{3.2}$$

The function $\phi(\theta)$ is a genuine periodic function. Furthermore, ϕ is an even function that is infinitely differentiable on the whole real line. We can expand ϕ in a Fourier cosine series that has very good convergence properties, i.e.,

$$\phi(\theta) = \tfrac{1}{2}\gamma_0 + \sum_{k=1}^{\infty} \gamma_k \cos k\theta. \tag{3.3}$$

By the addition formula for cosines, we have

$$\cos(k+1)\theta = \cos k\theta \cos\theta - \sin k\theta \sin\theta$$
$$\cos(k-1)\theta = \cos k\theta \cos\theta + \sin k\theta \sin\theta.$$

Adding these two equations, we obtain the recurrence relation

$$\cos(k+1)\theta = 2\cos\theta \cos k\theta - \cos(k-1)\theta. \tag{3.4}$$

Define

24

$$T_k(x) = \cos k\theta. \tag{3.5}$$

Then $T_k(x)$ satisfies the recurrence relation

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x). \tag{3.6}$$

Moreover, it follows from the definition in equation (3.5) that $T_0(x) = 1$ and $T_1(x) = x$. It follows from equation (3.6) by induction that $T_k(x)$ is a polynomial of degree k in x. The recurrence relation (3.6) can also be used to calculate $T_k(x)$ for a series of values of k. The polynomials $T_k(x)$ are called *Chebyshev polynomials of the first kind*. It is also easy to see from the recurrence relation that the highest power of x in $T_k(x)$ has the coefficient 2^{k-1}. We will now derive an expression for the remaining polynomial coefficients. Differentiating equation (3.5), we have

$$T_n'(x) = n\frac{\sin n\theta}{\sin \theta} \quad \text{and} \quad T_n''(x) = -n^2\frac{\cos n\theta}{\sin^2 \theta} + n\frac{\cos \theta \sin n\theta}{\sin^3 \theta}. \tag{3.7}$$

It follows that $T_n(x)$ satisfies the differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0. \tag{3.8}$$

Let us denote the coefficient of x^m in $T_n(x)$ by c_m^n, i.e.,

$$T_n(x) = \sum_{m=0}^{n} c_m^n x^m. \tag{3.9}$$

Substituting equation (3.9) into equation (3.8), we get

$$\sum_{m=0}^{n} m(m-1)c_m^n x^{m-2} + \sum_{m=0}^{n} (n^2 - m^2)c_m^n x^m = 0,$$

or equivalently

$$\sum_{m=0}^{n-2} (m+1)(m+2)c_{m+2}^n x^m + \sum_{m=0}^{n} (n^2 - m^2)c_m^n x^m = 0. \tag{3.10}$$

25

Equating the coefficients of the various powers of x to zero in equation (3.10), we get

$$c_{n-1}^n = 0 \tag{3.11a}$$

$$c_m^n = -\frac{(m+1)(m+2)}{n^2 - m^2} c_{m+2}^n \qquad m = 0, \dots, n-2. \tag{3.11b}$$

Since we know that $c_n^n = 2^{n-1}$, it follows from equations (3.11a) and (3.11b) that

$$c_{n-2k+1}^n = 0 \tag{3.12a}$$

$$c_{n-2k}^n = (-1)^m \frac{n(n-1)\dots(n-2m+1)}{m!(n-1)(n-2)\dots(n-m)} 2^{n-2m-1}. \tag{3.12b}$$

It should be noted that all the coefficients in $T_n(x)$ are integers.

In terms of the original variable x, the series expansion in equation (3.3) becomes

$$f(x) = \tfrac{1}{2}\gamma_0 + \sum_{k=1}^{\infty} \gamma_k T_k(x). \tag{3.13}$$

Thus, the function f can be expanded in a series of Chebyshev polynomials with a good rate of convergence. If we truncate this series at $k = N$, we obtain an N-th order polynomial approximation to f. The error in this approximation can be approximated by the first neglected term $\gamma_{N+1} T_{N+1}(x)$. Since the Chebyshev polynomials oscillate uniformly over the interval $[-1, 1]$, the error is very uniformly distributed over the interval. By way of contrast, polynomial approximations obtained by truncating a Taylor series expansion have a very small error near zero that increases rapidly away from zero.

Since the cosines $\cos k\theta$ satisfy the orthogonality relation

$$\int_0^\pi \cos k\theta \cos l\theta \, d\theta = 0 \qquad k \neq l, \tag{3.14}$$

it follows by the change of variable $x = \cos\theta$ that the Chebyshev polynomials satisfy the weighted orthogonality relation

$$\int_{-1}^1 \frac{T_k(x)T_l(x)}{\sqrt{1-x^2}} \, dx = 0 \qquad k \neq l. \tag{3.15}$$

The coefficients γ_k in the Chebyshev expansion are given by

$$\gamma_k = \frac{2}{\pi} \int_0^{\pi} f(\cos\theta)\cos k\theta \, d\theta = \frac{2}{\pi} \int_{-1}^{1} f(x)T_k(x)\frac{dx}{\sqrt{1-x^2}}. \tag{3.16}$$

In view of the definition of $T_n(x)$ in equation (3.5), many of the trigonometric identities have their counterpart in Chebyshev polynomials. For example, the trigonometric identity

$$\cos A \cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)] \tag{3.17}$$

leads to the Chebyshev polynomial identity

$$T_m(x)T_n(x) = \tfrac{1}{2}[T_{m+n}(x) + T_{|m-n|}(x)]. \tag{3.18}$$

Another useful identity for Chebyshev polynomials is

$$T_m\big(T_n(x)\big) = T_{mn}(x). \tag{3.19}$$

To see this, we can write the definition of T_m as

$$T_m(\hat{x}) = \cos m\hat{\theta} \quad \text{where} \quad \hat{\theta} = \cos^{-1}\hat{x}. \tag{3.20}$$

If we let

$$\hat{x} = T_n(x) = \cos n\theta \quad \text{where} \quad \theta = \cos^{-1} x, \tag{3.21}$$

then

$$\hat{\theta} = \cos^{-1}\cos n\theta = n\theta \quad \text{and} \quad T_m\big(T_n(x)\big) = \cos(mn\theta) = T_{mn}(x) \tag{3.22}$$

as was to be proved.

Sometimes it is more convenient to work with the interval $[0, 1]$ instead of the interval $[-1, 1]$. Suppose f is an infinitely smooth function on the interval $[0, 1]$. Let us make the change of variable

$$\cos\theta = 2x - 1 \quad \text{or} \quad x = \frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2} \tag{3.23}$$

and define

27

$$\phi(\theta) = f\left(\frac{1 + \cos\theta}{2}\right). \tag{3.24}$$

We can expand $\phi(\theta)$ in a fast convergent Fourier cosine series

$$\phi(\theta) = \tfrac{1}{2}\gamma_0 + \sum_{k=1}^{\infty} \gamma_k \cos k\theta. \tag{3.25}$$

We define

$$T_k^*(x) = \cos k\theta. \tag{3.26}$$

Then the series in equation (3.25) can be written

$$f(x) = \tfrac{1}{2}\gamma_0 + \sum_{k=1}^{\infty} \gamma_k T_k^*(x). \tag{3.27}$$

The coefficients in this series are given by

$$\gamma_k = \frac{2}{\pi} \int_0^\pi f\left(\frac{1+\cos\theta}{2}\right) \cos k\theta \, d\theta = \frac{4}{\pi} \int_0^1 f(x) T_k^*(x) \frac{dx}{\sqrt{1-x^2}}. \tag{3.28}$$

clearly, $T_0^*(x) = 1$ and $T_1^*(x) = 2x - 1$. Moreover, $T_n^*(x)$ satisfies the recursion relation

$$T_{k+1}^*(x) = 2(2x - 1)T_k^*(x) - T_{k-1}^*(x). \tag{3.29}$$

Thus, $T_k^*(x)$ is a polynomial of order k whose highest power of x has the multiplier 2^{2k-1}, $k > 0$. $T_k^*(x)$ is called a shifted Chebyshev polynomial. Clearly, T_k^* is related to T_k by $T_k^*(x) = T_k(2x - 1)$. Since $T_2(x) = 2x^2 - 1$, it follows from equation (3.19) that

$$T_k^*(x^2) = T_k(2x^2 - 1) = T_k(T_2(x)) = T_{2k}(x). \tag{3.30}$$

If $T_n^*(x)$ has the polynomial representation

$$T_n^*(x) = \sum_{m=0}^{n} d_m^n \, x^m, \tag{3.31}$$

then it follows from equation (3.30) that

28

$$T_n^*(x^2) = \sum_{m=0}^{n} d_m^n x^{2m} = T_{2n}(x) = \sum_{m=0}^{2n} c_m^{2n} x^m = \sum_{m=0}^{n} c_{2m}^{2n} x^{2m}. \tag{3.32}$$

Therefore, the polynomial coefficients are related by

$$d_m^n = c_{2m}^{2n}. \tag{3.33}$$

Other finite intervals can be reduced to $[-1, 1]$ or $[0, 1]$ by a linear change of variable.

3.2 Clenshaw's Summation Formula

We have seen that the trigonometric functions $\cos n\theta$ and the Chebyshev polynomials $T_n(x)$ satisfy two term recurrence relations. Clenshaw (1962) developed an efficient method for summing series involving functions that satisfy a two term recurrence relation [2]. Suppose we wish to compute the sum

$$f(x) = \sum_{k=0}^{N} c_k \phi_k(x) \tag{3.34}$$

where the functions $\phi_n(x)$ satisfy the recurrence relation

$$\phi_{n+1}(x) = \alpha_n(x)\phi_n(x) + \beta_n(x)\phi_{n-1}(x). \tag{3.35}$$

We calculate the sequence $y_N, \ldots, y_1$ by the recurrence

$$
\begin{aligned}
y_{N+2} &= y_{N+1} = 0 \\
y_k &= \alpha_k(x)y_{k+1} + \beta_{k+1}(x)y_{k+2} + c_k \qquad k = N, N-1, \ldots, 1.
\end{aligned}
\tag{3.36}
$$

Solving equation (3.36) for c_k and substituting the result into equation (3.34), we obtain

$$
\begin{aligned}
f(x) &= y_N \phi_N(x) \\
&= +[y_{N-1} - \alpha_{N-1}(x)y_N]\phi_{N-1}(x) \\
&= +[y_{N-2} - \alpha_{N-2}(x)y_{N-1} - \beta_{N-1}(x)y_N]\phi_{N-2}(x) \\
&= +[y_{N-3} - \alpha_{N-3}(x)y_{N-2} - \beta_{N-2}(x)y_{N-1}]\phi_{N-3}(x) \\
&= + \\
&= \vdots \\
&= + \\
&= +[y_2 - \alpha_2(x)y_3 - \beta_3(x)y_4]\phi_2(x) \\
&= +[y_1 - \alpha_1(x)y_2 - \beta_2(x)y_3]\phi_1(x) \\
&= +[c_0 + \beta_1(x)y_2 - \beta_1(x)y_2]\phi_0(x).
\end{aligned}
\tag{3.37}
$$

It can be seen from equation (3.37) that the terms multiplying each y_k ($k = N, \ldots, 2$) sum to zero in view of the recurrence relation (3.35). Thus, we are left with

$$f(x) = y_1 \phi_1(x) + [c_0 + \beta_1(x)y_2]\phi_0(x). \tag{3.38}$$

This summation method is a generalization of the nesting method commonly used to evaluate polynomials.

3.3 Telescoping Power Series

In general it is very difficult to get the coefficients of a Chebyshev series using the integral expressions in equations (3.16) and (3.28). In this, section we will describe another way to obtain approximate Chebyshev expansions. Suppose we are given a polynomial approximation to a function f defined on $[0, 1]$, i.e.,

$$f(x) \doteq a_0 + a_1 x + \cdots + a_N x^N. \tag{3.39}$$

This may have been obtained, for example, by truncating a Taylor series. In defining the shifted Chebyshev polynomials $T_n^*(x)$ we obtained the relation $x = \cos^2 \theta/2$. Using this relation, we can compute the powers of x as follows

$$
\begin{aligned}
x^n &= \cos^{2n} \frac{\theta}{2} = \left(\frac{e^{i\theta/2} + e^{-i\theta/2}}{2} \right)^{2n} \\
&= \frac{2}{4^n} \left[\cos n\theta + \binom{2n}{1} \cos(n-1)\theta + \cdots + \binom{2n}{n} \frac{1}{2} \right] \\
&= \frac{2}{4^n} \left[T_n^*(x) + \binom{2n}{1} T_{n-1}^*(x) + \cdots + \frac{1}{2} \binom{2n}{n} T_0^*(x) \right].
\end{aligned} \tag{3.40}
$$

The first six powers of x are given by

$$1 = T_0^*(x)$$
$$x = \frac{T_0^*(x) + T_1^*(x)}{2}$$
$$x^2 = \frac{3T_0^*(x) + 4T_1^*(x) + T_2^*(x)}{8}$$
$$x^3 = \frac{10T_0^*(x) + 15T_1^*(x) + 6T_2^*(x) + T_3^*(x)}{32}$$
$$x^4 = \frac{35T_0^*(x) + 56T_1^*(x) + 28T_2^*(x) + 8T_3^*(x) + T_4^*(x)}{128}$$
$$x^5 = \frac{126T_0^*(x) + 210T_1^*(x) + 120T_2^*(x) + 45T_3^*(x) + 10T_4^*(x) + T_5^*(x)}{512}$$
$$x^6 = \frac{462T_0^*(x) + 792T_1^*(x) + 495T_2^*(x) + 220T_3^*(x) + 66T_4^*(x) + 12T_5^*(x) + T_6^*(x)}{2048}$$

Substituting the expressions for the various powers of x into equation (3.39) and rearranging terms, we can obtain an expansion of $f(x)$ in terms of the shifted Chebyshev polynomials. The higher order terms in this expansion often have small coefficients and can be neglected with little loss of

accuracy. In this way we can obtain a lower order polynomial approximation with almost the same accuracy.

Example 4. Let $f(x)$ be defined by

$$f(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6. \tag{3.41}$$

You might recognize this as the truncated Taylor series expansion of $1/(1 + x)$. Substituting the expressions for the powers of x in terms of the shifted Chebyshev polynomials $T_n^*(x)$ into the above polynomial and rearranging terms, we obtain

$$f(x) = a_0 + a_1 T_1^*(x) + \cdots + a_6 T_6^*(x) \tag{3.42}$$

where the coefficients a_n are given by

n	a_n
0	0.81542969
1	-0.05468750
2	0.16357422
3	0.05078125
4	0.02050781
5	0.00390625
6	0.00048828

Figure 3.1 shows the effect of dropping x^6 in equation (3.41) and Figure 3.2 shows the effect of dropping the $T_6^*(x)$ term in equation (3.42).

Clearly, dropping x^6 in equation (3.41) leads to large errors, whereas dropping the $T_6^*(x)$ term in equation (3.42) makes very little difference. Figure 3.3 shows the effect of dropping both the $T_5^*(x)$ and the $T_6^*(x)$ terms in equation (3.42). Again the resulting approximation is quite good. Figure 3.4 shows the effect of dropping the $T_4^*(x)$, the $T_5^*(x)$, and the $T_6^*(x)$ terms in equation (3.42). In this case we can see some deviation of the approximation.

Although this telescoping method allows us to express $f(x)$ in terms of lower order polynomials, we do not increase the accuracy. If the polynomial $f(x)$ is an approximation to some function, the Chebyshev approximations obtained in this way will have no better accuracy than the original polynomial. In this example the polynomial $f(x)$ is a poor approximation to $1/(1 + x)$.
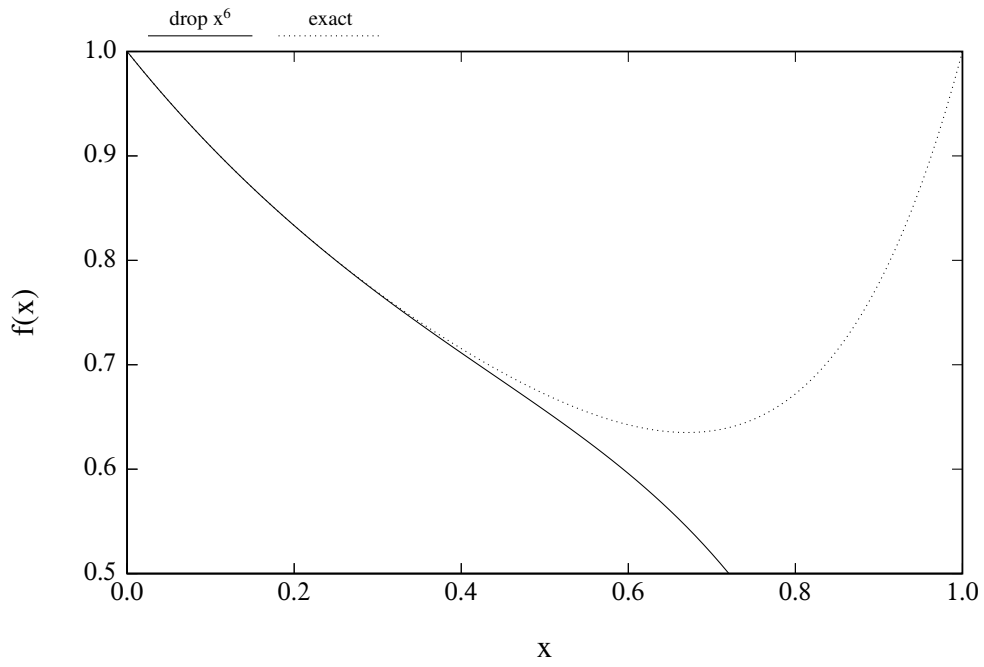
Figure 3.1: The effect of dropping x^6 in polynomial defining $f(x)$
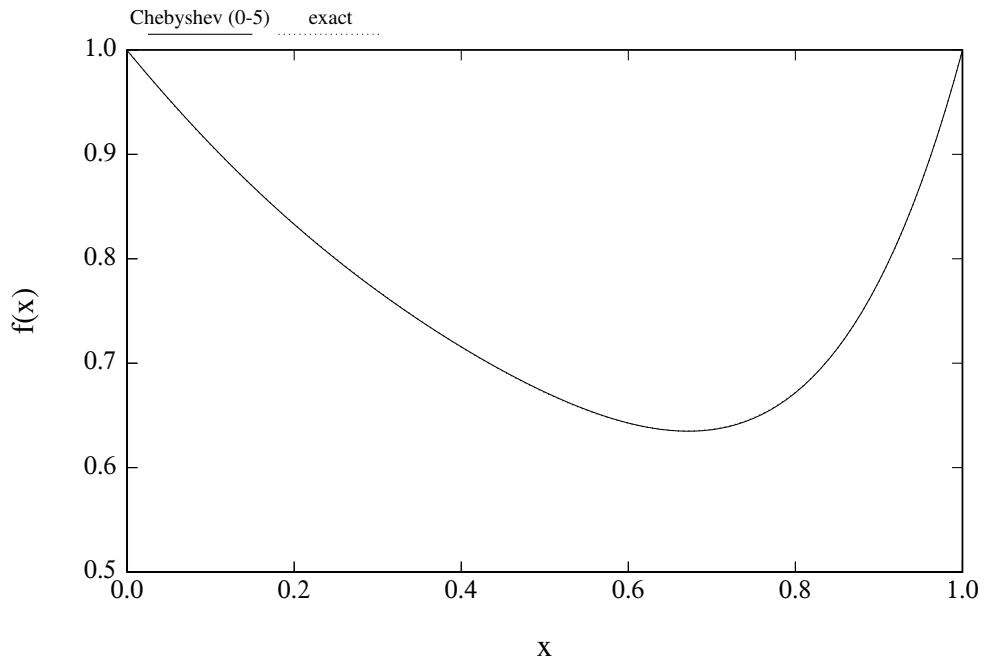

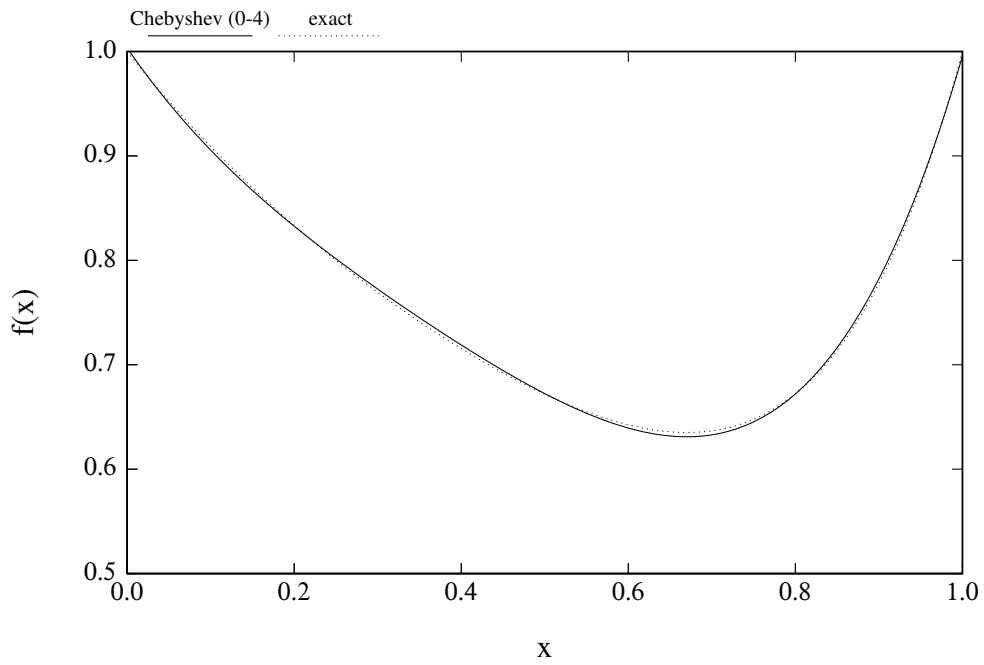Figure 3.2: The effect of dropping $T_6^*(x)$ term in expansion of $f(x)$.

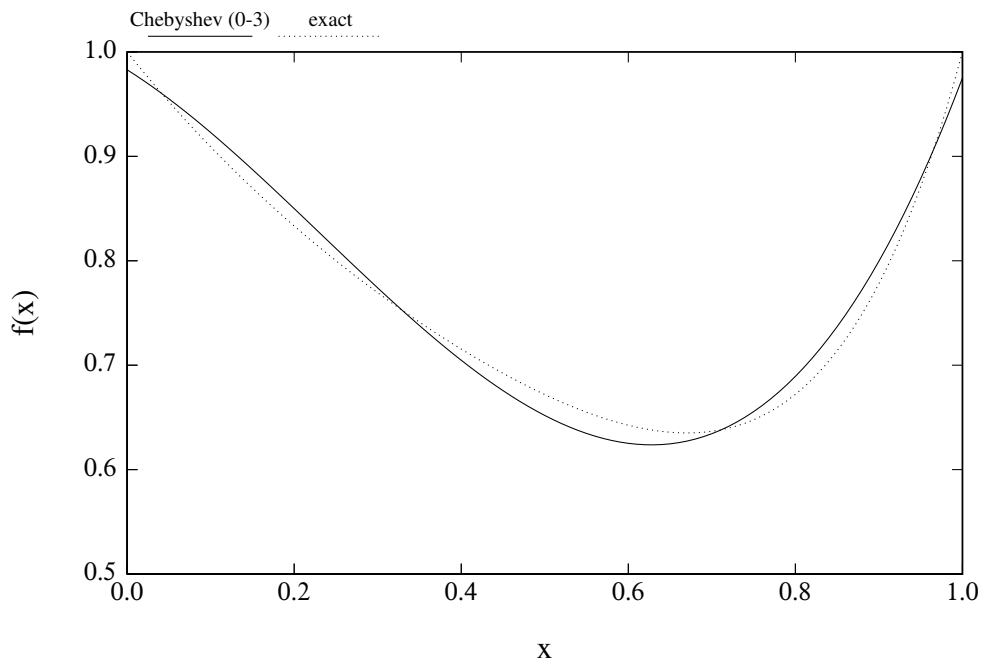Figure 3.3: The effect of dropping $T_5^*(x)$ and $T_6^*(x)$ terms in expansion of $f(x)$.

Figure 3.4: The effect of dropping $T_4^*(x)$, $T_5^*(x)$, and $T_6^*(x)$ terms in expansion of $f(x)$.

3.4 The Lanczos Tau Method

Lanczos developed the τ method to obtain polynomial approximations to functions $y(x)$ that are solutions of a linear differential or algebraic equation having coefficients that are rational functions of x. By multiplying through by an appropriate polynomial, differential or algebraic equations of this type can be reduced to ones having polynomial coefficients. The Tau Method makes use of Chebyshev polynomials to obtain an even error distribution over the interval of interest. It is applicable to many of the standard functions used in mathematical physics. We will illustrate the method with an example. Consider the function $f(x) = 1/(1+x)$ on the interval $[0, 2]$. Since the shifted Chebyshev polynomials are defined on $[0, 1]$, we will instead work with the scaled function $y(x) = f(2x) = 1/(1+2x)$ on $[0, 1]$. This function satisfies the differential equation

$$(1 + 2x)y' + 2y = 0 \tag{3.43}$$

on the interval $[0, 1]$ with the initial condition $y(0) = 1$. One of the standard ways of solving equations of this type is to substitute a power series representation of y into the differential equation and to obtain a recurrence relation for the power series coefficients. Assume y has a power series representation

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{3.44}$$

Substituting this power series expression into equation (3.43), we obtain

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

or equivalently

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0.. \tag{3.45}$$

Equating the coefficients of each power of x to zero, we obtain

$$a_1 + 2a_0 = 0 \tag{3.46a}$$
$$(n+1)a_{n+1} + 2(n+1)a_n = 0 \qquad n = 1, 2, \ldots \tag{3.46b}$$

Since $y(0) = 1$, it follows that $a_0 = 1$. The remaining coefficients a_n can be computed using the recurrence relations in equation (3.46). This leads to the power series representation

$$y = \frac{1}{1 + 2x} = 1 - 2x + (2x)^2 - (2x)^3 + \dots. \qquad (3.47)$$

Suppose we truncate the power series and consider the polynomial approximation

$$y(x) = \sum_{n=0}^{N} a_n x^n. \qquad (3.48)$$

Substituting this polynomial into the differential equation, we get

$$\sum_{n=1}^{N} n a_n x^{n-1} + 2 \sum_{n=1}^{N} n a_n x^n + \sum_{n=0}^{N} a_n x^n = 0$$

or equivalently

$$\sum_{n=0}^{N-1} (n+1) a_{n+1} x^{n-1} + 2 \sum_{n=1}^{N} n a_n x^n + 2 \sum_{n=0}^{N} a_n x^n = 0. \qquad (3.49)$$

Equating the coefficients of each power of x to zero, we obtain

$$a_1 + 2a_0 = 0 \qquad (3.50a)$$
$$(n+1)a_{n+1} + 2(n+1)a_n = 0 \qquad n = 1, 2, \dots, N-1 \qquad (3.50b)$$
$$2(N+1)a_N = 0. \qquad (3.50c)$$

We obviously can't require that equation (3.50c) hold for this would force all the a_n to be zero. Since this equation comes from setting the coefficient of the N-th power of x to zero, Lanczos had the idea of putting an error term on the right hand side of the differential equation that has the form τx^N, i.e.,

$$(1 + 2x)y' + 2y = \tau x^N. \qquad (3.51)$$

If we substitute our polynomial into this equation, we obtain

$$a_1 + 2a_0 = 0 \qquad (3.52a)$$
$$(n+1)a_{n+1} + 2(n+1)a_n = 0 \qquad n = 1, 2, \dots, N-1 \qquad (3.52b)$$
$$2(N+1)a_N = \tau. \qquad (3.52c)$$

Starting with $a_0 = 1$, we can compute $a_1, \ldots, a_N$ using equations (3.52a)–(3.52b). The error coefficient τ can then be computed from equation (3.52c). This gives the truncated Taylor series expansion of $y(x)$ and a τ value of $2(N+1)(-2)^N$. This τ value is quite large since we are going past the radius of convergence of the Taylor series.

The problem with an error term of the form τx^N is that the error is very small near $x = 0$, but increases rapidly near $x = 1$. Lanczos noted that we could use any polynomial of degree N in place of x^N for the error term. A polynomial with evenly distributed values over $[0, 1]$ is the Chebyshev polynomial $T_N^*(x)$. With this choice, the approximate differential equation becomes

$$(1 + 2x)y' + 2y = \tau T_N^*(x). \tag{3.53}$$

If

$$T_N^*(x) = \sum_{n=0}^{N} c_n^N x^n, \tag{3.54}$$

then substituting the polynomial expression for y into equation (3.53) gives

$$a_1 + 2a_0 = \tau c_0^N \tag{3.55a}$$

$$(n+1)a_{n+1} + 2(n+1)a_n = \tau c_n^N \qquad n = 1, 2, \ldots, N-1 \tag{3.55b}$$

$$2(N+1)a_N = \tau c_N^N. \tag{3.55c}$$

We can obtain a_N as a multiple of τ from equation (3.55c). Starting with this expression for a_N we can obtain $a_{N-1}, \ldots, a_1$ as multiples of τ from equation (3.55b). Since $a_0 = 1$ and a_1 is now a known multiple of τ, we can solve equation (3.55a) for τ. Having τ, all the a_n can then be determined. For $N = 6$ we obtain an approximation

$$y(x) \doteq a_0 + a_1 x + \cdots + a_6 x^6 \tag{3.56}$$

where the coefficients are given by

and $\tau = 0.0061734$. This approximation along with the exact expression are plotted in figure 3.5. In this plot the x values have been scaled to the interval $[0, 2]$. The relative error is plotted in figure 3.6. Notice that we have obtained a good polynomial approximation of $1/(1 + x)$ beyond the radius of convergence of the Taylor series expansion.

The technique we have illustrated is known as Lanczos' tau method. It can be used to obtain polynomial approximations to many of the important functions in mathematical physics.

In some problems it will be necessary to use more than one error term with corresponding τ factors. Consider, for example, the initial-value problem
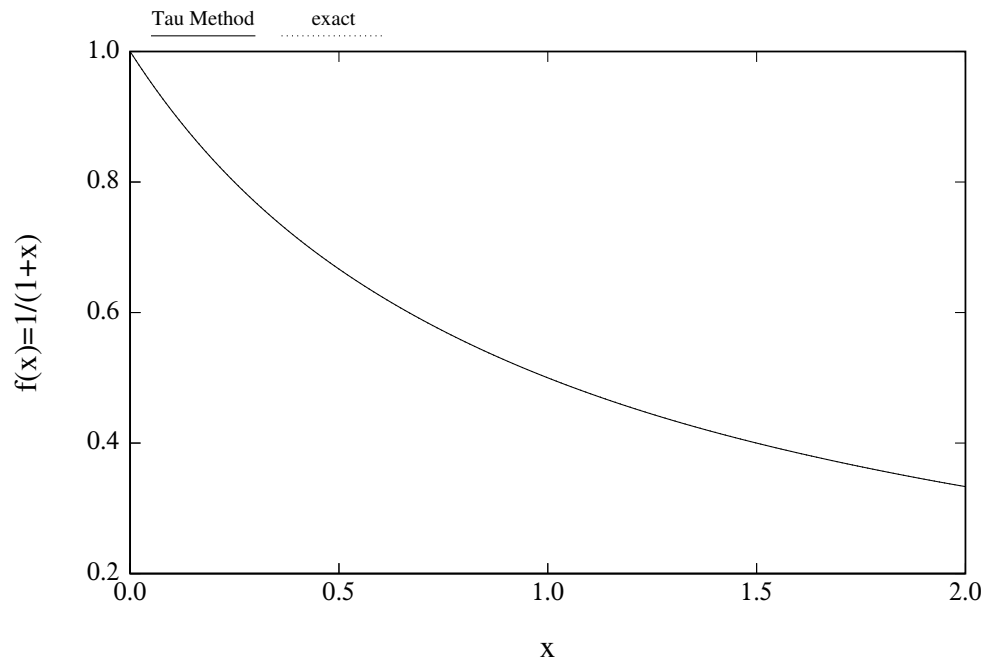
Figure 3.5: Tau method approximation to $y(x) = 1/(1 + x)$
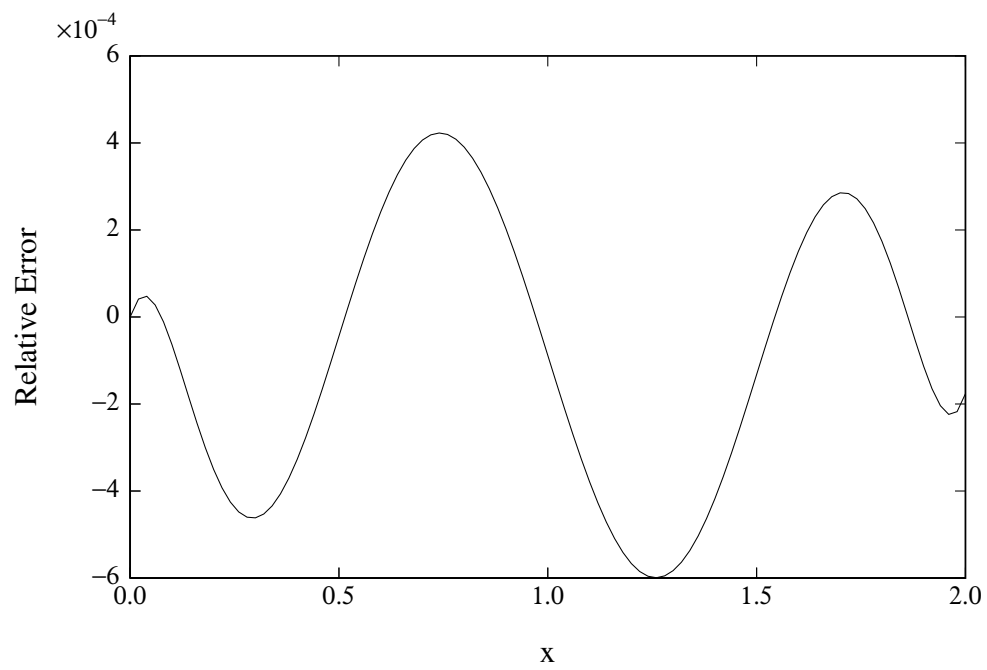
Figure 3.6: Relative error for tau method approximation of $y(x) = 1/(1 + x)$

n	a_n
0	1.0000000
1	-1.9938266
2	3.7654114
3	-5.8022753
4	6.0731987
5	-3.6123115
6	0.9030779

$$(1 + x^2)y' + y = 0 \qquad \text{with} \qquad y(0) = 1. \tag{3.57}$$

We will approximate y by a polynomial in x, i.e.,

$$y(x) = \sum_{n=0}^{N} a_n x^n. \tag{3.58}$$

Substituting this polynomial expression into equation (3.57), we get

$$\sum_{n=1}^{N} n a_n x^{n-1} + \sum_{n=1}^{N} n a_n x^{n+1} + \sum_{n=0}^{N} a_n x^n = 0 \tag{3.59}$$

or equivalently

$$\sum_{n=0}^{N-1} (n + 1)a_{n+1}x^n + \sum_{n=2}^{N+1} (n - 1)a_{n-1}x^n + \sum_{n=0}^{N} a_n x^n = 0. \tag{3.60}$$

Equating the coefficients of each power of x to zero, we obtain

$$a_1 + a_0 = 0 \tag{3.61a}$$
$$2a_2 + a_1 = 0 \tag{3.61b}$$
$$(n + 1)a_{n+1} + (n - 1)a_{n-1} + a_n = 0 \qquad n = 1, 2, \ldots, N - 1 \tag{3.61c}$$
$$(N - 1)a_{N-1} + a_N = 0 \tag{3.61d}$$
$$Na_N = 0. \tag{3.61e}$$

We obviously can't require that equations (3.61d) and (3.61e) hold, since that would force all of the a_n to be zero. Since equations (3.61d) and (3.61e) come from equating coefficients of x^N and x^{N+1}, we could put an error term of the form

$$\tau_1 T_N^*(x) + \tau_2 T_{N+1}^*(x)$$

on the right-hand-side of the differential equation (3.57). However, it is simpler and almost as effective to use an error term of the form

$$T_N^*(x)(\tau_1 + \tau_2 x).$$

Thus, the modified problem becomes

$$(1 + x^2)y' + y = T_N^*(x)(\tau_1 + \tau_2 x) \qquad \text{with} \qquad y(0) = 1. \qquad (3.62)$$

If $T_N^*(x)$ has the form shown in equation (3.54), then the coefficients a_n must satisfy

$$a_1 + a_0 = \tau_1 c_0^N \qquad (3.63a)$$
$$2a_2 + a_1 = \tau_1 c_1^N + \tau_2 c_0^N \qquad (3.63b)$$
$$(n+1)a_{n+1} + (n-1)a_{n-1} + a_n = \tau_1 c_n^N + \tau_2 c_{n-1}^N \qquad n = 1, 2, \ldots, N-1 \qquad (3.63c)$$
$$(N-1)a_{N-1} + a_N = \tau_1 c_N^N + \tau_2 c_{N-1}^N \qquad (3.63d)$$
$$N a_N = \tau_2 c_N^N. \qquad (3.63e)$$

Using equations (3.63d) and (3.63e), we can express a_N and a_{N-1} as linear combinations of τ_1 and τ_2. Equation (3.63c) can then used to express $a_{N-2}, \ldots, a_1$ as linear combinations of τ_1 and τ_2. Since $a_0 = 1$, equations (3.63a) and (3.63b) become a system of two equations that can be solved for τ_1 and τ_2. Having τ_1 and τ_2, we can then go back and get $a_1, \ldots, a_N$. Often the number of tau factors that are needed can be reduced by a change of variable. It is rare when more than two tau factors are needed. The tau method can also be applied to algebraic equations such as

$$(x + 1)y = 1.$$

3.5 Operational Approach to Tau Method

Following Lanczos' introduction of the Tau Method, a number of authors have presented modifications and alternative approaches [1, 5, 9, 10]. Clenshaw, instead of using expansions in terms of powers, developed solutions in terms of sums of Chebyshev polynomials directly [1]. He expanded the highest order derivative in terms of Chebyshev polynomials and then used recursion relations for integrals of Chebyshev polynomials to generate lower order derivatives. In most cases this method is not simple to implement and will not be discussed here. Ortiz, a student of Lanczos, developed an operational approach to the tau method that greatly simplified its implementation in software [10]. I will summarize the operational approach in this section. It is called an operational method since it replaces the differential equation problem by a problem in matrix algebra.

A differential equation with polynomial coefficients can be written as

$$Dy = \sum_{i=0}^{v} p_i(x) \frac{d^i y}{dx^i} = 0. \tag{3.64}$$

We seek to approximate the solution of this differential equation along with the following set of auxiliary conditions

$$f_k(y) = s_k \qquad k = 1, \ldots, v. \tag{3.65}$$

Here each f_k is a linear functional (linear function mapping functions into real numbers) operating on y. The usual boundary and initial-value conditions can be put in this form. We will attempt to solve the differential equation problem in terms of sums involving a set of independent polynomials $v_0(x), v_1(x), \ldots$ where the polynomial $v_k(x)$ has order k. Possible choices for these polynomials are the Chebyshev polynomials or the Legendre Polynomials. Let y_n denote the approximate solution

$$y_n(x) = a_0 v_0(x) + a_1 v_1(x) + \cdots + a_n v_n(x). \tag{3.66}$$

Let N be the highest power of x occurring in $Dy_n(x)$. If $N < n$, we set $N = n$. For example, substitution of the polynomial $y_n(x)$ of order n into the differential operator $x^3 y' - 2y$, results in a polynomial of order $n + 2$. Thus, in this case, $N = n + 2$. In the material that follows we will only consider polynomials of order less than or equal to N. The reason we defined N so that $N \geq n$ is that we will require that the polynomial $y_n(x)$ of order n satisfy the auxiliary conditions of equation (3.65) as well as approximating the solution of the differential equation. Equation (3.66) can be written in the matrix form

$$y_n(x) = \boldsymbol{a}_n^T \boldsymbol{v}, \tag{3.67}$$

where $\boldsymbol{a}_n$ and $\boldsymbol{v}$ are the $(N + 1)$-vectors given by

$$\boldsymbol{a}_n^T = (a_0, a_1, \ldots, a_n, 0, \ldots, 0) \quad \text{and} \quad \boldsymbol{v}^T = \big(v_0(x), v_1(x), \ldots, v_N(x)\big) \qquad (3.68)$$

and the superscript T denotes matrix transpose. The next step is to represent differentiation of polynomials and multiplication of polynomials by powers of x as matrix operations. If $p(x)$ is the polynomial

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n, \qquad (3.69)$$

then we can write this polynomial in the matrix form

$$p(x) = \boldsymbol{\alpha}^T \boldsymbol{x}, \qquad (3.70)$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{x}$ are the $N + 1$ vectors given by

$$\boldsymbol{\alpha}^T = (\alpha_0, \alpha_1, \ldots, \alpha_n, 0, \ldots, 0) \quad \text{and} \quad \boldsymbol{x}^T = (1, x, x^2, \ldots, x^N). \qquad (3.71)$$

It follows that

$$\begin{aligned}
\frac{d}{dx} p(x) &= \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \cdots + n\alpha_n x^{n-1} \\
&= \boldsymbol{\alpha}^T \boldsymbol{\Delta} \boldsymbol{x},
\end{aligned} \qquad (3.72)$$

where $\boldsymbol{\Delta}$ is the matrix

$$\boldsymbol{\Delta} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & N & 0 \end{pmatrix}. \qquad (3.73)$$

It follows by induction that

$$\frac{d^k}{dx^k} p(x) = \boldsymbol{\alpha}^T \boldsymbol{\Delta}^k \boldsymbol{x}. \qquad (3.74)$$

Similarly

43

$$xp(x) = \alpha_0 x + \alpha_1 x^2 + \cdots + \alpha_n x^{n+1}$$
$$= \alpha^T M x, \tag{3.75}$$

where M is the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}. \tag{3.76}$$

Again, it follows by induction that

$$x^k p(x) = \alpha^T M^k x. \tag{3.77}$$

The operations of multiplication by x and differentiation can be combined as follows

$$x^i \frac{d^j}{dx^j} p(x) = x^i \alpha^T \Delta^j x = \alpha^T \Delta^j M^i x. \tag{3.78}$$

We will now apply the above results to the approximation $y_n(x)$. The polynomials $v_0(x), \ldots, v_N(x)$ involved in the definition of y_n are independent and the polynomial $v_k(x)$ has order k. Therefore, there is a nonsingular lower triangular matrix V such that

$$v = Vx. \tag{3.79}$$

For the Chebyshev polynomials the matrix V is given by

$$V = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ -1 & 0 & 2 & & \\ 0 & -3 & 0 & 4 & \\ 1 & 0 & -8 & 0 & 8 \\ & \cdots & \cdots & \cdots & \end{pmatrix}. \tag{3.80}$$

Let us denote the m-th row of V by $V_{m\cdot}$ and let $V_{\cdot m}$ denote the m-th column of V. The polynomials $v_k(x)$ can be written as

$$v_k(x) = (V_{k\cdot})x \qquad k = 0, \ldots, N. \tag{3.81}$$

Therefore, it follows that

$$
\begin{aligned}
x^i \frac{d^j}{dx^j} y_n(x) &= a_0 x^i \frac{d^j}{dx^j} v_0(x) + \cdots + a_n x^i \frac{d^j}{dx^j} v_N(x) \\
&= a_0 V_{0\cdot} \boldsymbol{\Delta}^j M^i x + \cdots + a_n V_{n\cdot} \boldsymbol{\Delta}^j M^i x \\
&= \boldsymbol{a}^T V \boldsymbol{\Delta}^j M^i x = \boldsymbol{a}^T V \boldsymbol{\Delta}^j M^i V^{-1} \boldsymbol{v}.
\end{aligned} \tag{3.82}
$$

In view of equation (3.82), we have

$$Dy_n = \boldsymbol{a}^T V \sum_{i=0}^{\nu} \boldsymbol{\Delta}^i p_i(M) V^{-1} \boldsymbol{v}. \tag{3.83}$$

If we define

$$\boldsymbol{\Pi} = V \sum_{i=0}^{\nu} \boldsymbol{\Delta}^i p_i(M) V^{-1},$$

then equation (3.83) can be written

$$
\begin{aligned}
Dy_n &= \boldsymbol{a}^T \boldsymbol{\Pi} \boldsymbol{v} \\
&= (\boldsymbol{a}^T \boldsymbol{\Pi}_{\cdot 0}) v_0(x) + \cdots + (\boldsymbol{a}^T \boldsymbol{\Pi}_{\cdot N}) v_N(x).
\end{aligned} \tag{3.84}
$$

We will require that y_n satisfy the ν auxiliary conditions given in equation (3.65). Thus,

$$f_k(y_n) = f_k \Big(\sum_{i=0}^{n} a_i v_i \Big) = \sum_{i=0}^{n} a_i f_k(v_i) = s_k \qquad k = 0, \ldots, \nu. \tag{3.85}$$

This gives us ν linear equations for the unknown coefficients $a_0, \ldots, a_n$. We need $n+1$ equations to determine these coefficients. To get the remaining $n - \nu + 1$ equations we set

$$\boldsymbol{a}^T \boldsymbol{\Pi}_{\cdot 0} = \boldsymbol{a}^T \boldsymbol{\Pi}_{\cdot 1} = \cdots = \boldsymbol{a}^T \boldsymbol{\Pi}_{\cdot n-\nu} = 0 \tag{3.86}$$

in equation (3.84). Equations (3.85) and (3.86) give us $n + 1$ linear equations that completely determine the coefficients $a_0, \ldots, a_n$. With this choice of coefficients y_n satisfies

$$Dy_n = (\boldsymbol{a}^T \boldsymbol{\Pi}_{.n-\nu+1})v_{n-\nu+1}(x) + \cdots + (\boldsymbol{a}^T \boldsymbol{\Pi}_{.N})v_N(x). \tag{3.87}$$

The coefficients multiplying the polynomials $v_{n-\nu+1}(x), \dots, v_N(x)$ on the right-hand-side of equation (3.87) are Lanczos' tau values. This operational approach is quite easy to implement on a computer.

3.6   Generalized Tau Method

The tau method is only directly applicable if the coefficients of the given differential or algebraic equation are rational functions of x. Consider a general linear differential equation

$$Dy = 0 \tag{3.88}$$

on the interval [0,1]. We want to approximate the solution of this equation by a polynomial y_p of the form

$$y_p(x) = \sum_{n=0}^{p} a_n x^n. \tag{3.89}$$

As in the standard tau method we replace the right hand of equation (3.88) by an error term of the form

$$T_N^*(x)(\tau_1 + \tau_2 x + \cdots + \tau_m x^{m-1}).$$

Thus, equation (3.88) is replaced by

$$Dy = T_N^*(x)(\tau_1 + \tau_2 x + \cdots + \tau_m x^{m-1}). \tag{3.90}$$

The polynomial y_p has $p+1$ unknown coefficients $a_0, a_1, \ldots, a_p$. If there are ν initial or boundary conditions given, then there are $p + 1 - \nu$ free parameters at our disposal. We set $N = p + 1 - \nu$ in equation (3.90). We substitute y_p into equation (3.90) and require that the resulting equation

$$Dy_p = T_N^*(x)(\tau_1 + \tau_2 x + \cdots + \tau_m x^{m-1}) \tag{3.91}$$

be satisfied at the N zeroes of $T_N^*(x)$. The right-hand-side of equation (3.91) is zero at these N evaluation points. This gives us N linear equations in the coefficients $a_0, a_1, \ldots, a_p$ that along with the ν initial or boundary conditions provides a set of $p + 1$ linear equations to be solved for the $p + 1$ coefficients. The zeroes x_k of $T_N^*(x)$ are given by

$$x_k = \frac{1 + \cos[(2k - 1)\pi/(2N)]}{2} \qquad k = 1, 2, \ldots, N. \tag{3.92}$$

4   Fourier Integral

The Fourier integral represents a function h (defined on the real line) by

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft}\, df \tag{4.1}$$

where

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\, dt. \tag{4.2}$$

The functions h and H are called a Fourier transform pair. The Fourier integral exists if h is absolutely integrable on the real line and is of bounded variation on every finite subinterval.

4.1   Properties of Fourier Transforms

It is easily shown that the Fourier transform has the properties shown in the table below:

Table 4.1: Properties of Fourier Transforms

If ...	then ...
$h(t)$ is real	$H(-f) = \overline{H(f)}$
$H(-f) = \overline{H(f)}$	$h(t)$ is real
$h(t)$ is imaginary	$H(-f) = -\overline{H(f)}$
$H(-f) = -\overline{H(f)}$	$h(t)$ is imaginary
$h(t)$ is even	$H(f)$ is even
$H(f)$ is even	$h(t)$ is even
$h(t)$ is odd	$H(f)$ is odd
$H(f)$ is odd	$h(t)$ is odd
$h(t)$ is real and even	$H(F)$ is real and even
$H(F)$ is real and even	$h(t)$ is real and even
$h(t)$ is real and odd	$H(F)$ is imaginary and odd
$H(F)$ is imaginary and odd	$h(t)$ is real and odd

Continued on next page

48

If ...	then ...	
$h(t)$ is imaginary and even	$H(F)$ is imaginary and even	
$H(F)$ is imaginary and even	$h(t)$ is imaginary and even	
$h(t)$ is imaginary and odd	$H(F)$ is real and odd	
$H(F)$ is real and odd	$h(t)$ is imaginary and odd	
$h(t) = df(t)/dt$	$H(f) = i2\pi f\, F(f)$	differentiation
$\hat{h}(t) = h(at)$	$\hat{H}(f) = H(f/a)/\,|a|$	time scaling
$\hat{H}(f) = H(bf)$	$\hat{h}(t) = h(t/b)/\,|b|$	frequency scaling
$\hat{h}(t) = h(t - t_0)$	$\hat{H}(f) = e^{-i2\pi f t_0} H(f)$	time shifting
$\hat{H}(f) = H(f - f_0)$	$\hat{h}(t) = e^{i2\pi f_0 t} h(t)$	frequency shifting
$h(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\, d\tau$	H(f)=F(f)G(f)	convolution to product
H(f)=F(f)G(f)	$h(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\, d\tau$	product to convolution

In these formulas, a bar over a quantity indicates complex conjugate.

4.2   Relation of Fourier Transform to Fourier Series

The Fourier transform of h is given by

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\, dt. \tag{4.3}$$

On the interval $[-L, L]$, h has the Fourier series expansion

$$h(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi t/L} \tag{4.4}$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} h(t)e^{-i\pi nt/L}\, dt. \tag{4.5}$$

Comparing equations (4.3) and (4.5), we see that

49

$$La_n = H\left(\frac{n}{2L}\right) - \int_L^\infty h(t)e^{-i\pi nt/L}\,dt - \int_{-\infty}^{-L} h(t)e^{-i\pi nt/L}\,dt. \tag{4.6}$$

If h is absolutely integrable, the integrals on the right-hand-side of equation (4.6) can be made arbitrarily small for sufficiently large L. It is in this sense that the Fourier series coefficients a_n can be used to approximate the Fourier transform at the frequencies $\frac{n}{2L}$.

4.3   Asymptotic Behavior

In physical problems we usually have $h(t) = 0$ for $t < 0$. In this case

$$H(f) = \int_0^\infty h(t)e^{-i2\pi ft}\,dt. \tag{4.7}$$

If h is sufficiently smooth and converges sufficiently rapidly to zero as $t \to \infty$, we can integrate equation (4.7) by parts three times to obtain

$$H(f) = \frac{h(0)}{i2\pi f} + \frac{h'(0)}{(i2\pi f)^2} + \frac{h''(0)}{(i2\pi f)^3} + \frac{1}{(i2\pi f)^3}\int_0^\infty h'''(t)e^{-i2\pi ft}\,dt. \tag{4.8}$$

We can see from equation (4.8) that the behavior of $H(f)$ for large f is related to the behavior of $h(t)$ for small t. Thus, high frequency approximations can often be used to obtain small time approximations and vice versa.

4.4 Input-Output Relations

In many linear problems it is easier to solve the problem in the frequency domain than in the time domain. This is due largely to the fact that time derivatives become algebraic expressions in the frequency domain. The inverse Fourier transform can be used to relate the solution in the time domain to the solution in the frequency domain. In such problems we are often able to relate an output frequency function $G(f)$ to and input frequency function $F(f)$ by a linear relation

$$G(f) = R(f)F(f). \tag{4.9}$$

The function $R(f)$ is called a *transfer function*. Relations of this type are common in electric network problems. The Fourier transform of this relation is given by

$$g(t) = \int_{-\infty}^{\infty} K(t - \tau) f(\tau) \, d\tau \tag{4.10}$$

where

$$K(t) = \int_{-\infty}^{\infty} R(f) e^{i2\pi ft} \, dt. \tag{4.11}$$

The function $K(t)$ is called the *impulse response* since it is the output obtained from a delta function input. Since the output at time t can't depend on the input at times greater than t, it follows that

$$K(t) = 0 \qquad \text{for } t < 0. \tag{4.12}$$

Thus,

$$g(t) = \int_{-\infty}^{t} K(t - \tau) f(\tau) \, d\tau. \tag{4.13}$$

If the input $f(t)$ is zero for $t < 0$, then

$$g(t) = \int_{0}^{t} K(t - \tau) f(\tau) \, d\tau. \tag{4.14}$$

Taking the inverse transform of equation (4.11) and using the relation in equation (4.12), we obtain

$$R(f) = \int_{0}^{\infty} K(t) e^{-i2\pi ft} \, dt. \tag{4.15}$$

Let A and B be the real and imaginary parts of R, i.e.,

$$R(f) = A(f) + iB(f). \tag{4.16}$$

Then, it follows from equation (4.15) that

$$A(f) = \int_0^\infty K(t) \cos(2\pi ft) \, dt \tag{4.17}$$

and

$$B(f) = -\int_0^\infty K(t) \sin(2\pi ft) \, dt. \tag{4.18}$$

It is clear from equations (4.17) and (4.18) that $A(f)$ is even $[A(-f) = A(f)]$ and $B(f)$ is odd $[B(-f) = -B(f)]$. Writing equation (4.11) in terms of A and B and using the symmetry properties of A and B, we obtain

$$K(t) = 2 \int_0^\infty A(f) \cos(2\pi ft) \, dt - 2 \int_0^\infty B(f) \sin(2\pi ft) \, dt. \tag{4.19}$$

It follows from equation (4.12) that

$$\int_0^\infty A(f) \cos(2\pi ft) \, dt - \int_0^\infty B(f) \sin(2\pi ft) \, dt = 0 \qquad \text{for } t < 0. \tag{4.20}$$

Replacing t by $-t$, we obtain the relation

$$\int_0^\infty A(f) \cos(2\pi ft) \, dt + \int_0^\infty B(f) \sin(2\pi ft) \, dt = 0 \qquad \text{for } t > 0. \tag{4.21}$$

Combining equations (4.19) and (4.21), we get

$$K(t) = 4 \int_0^\infty A(f) \cos(2\pi ft) \, dt = -4 \int_0^\infty B(f) \sin(2\pi ft) \, dt. \tag{4.22}$$

Thus, the impulse response can be obtained from either the real or the imaginary part of the transfer function R. However, the convergence properties of the integrals in equation (4.22) can be very different.

4.4.1 Calculation of Impulse Responses

The impulse response is generally zero for $t < 0$, jumps to a finite value at $t = 0$, and decays rapidly to zero as $t \to \infty$. In order to approximate the impulse response it will be necessary to truncate the response at some $T > 0$. We will choose T so that $K(T) \doteq 0$. If we expand K in a Fourier series on the interval $[0, T]$, the series will be slowly convergent [like 1/n] because of the jump at $t = 0$. Since the Fourier series coefficients approximate the Fourier transform, the Fourier transform converges slowly [like 1/f]. In many problems an asymptotic analysis gives us the value K_0 of K at $t = 0$. We can use this information to improve the convergence. Define

$$\hat{K}(t) = K(t) - K_0 e^{-\alpha t} \qquad \text{for } t \geq 0 \tag{4.23}$$

where α is chosen so that the exponential term is approximately zero at $t = T$. The function $\hat{K}$ is zero at $t = 0$ and is approximately zero at $t = T$. Thus, we can obtain $1/n^3$ convergence by extending $\hat{K}$ to $[-T, T]$ as an odd function. The Fourier transform $\hat{R}$ of the extended function $\hat{K}$ is given by

$$
\begin{aligned}
\hat{R}(f) &= \int_{-\infty}^{\infty} \hat{K}(t) e^{-i2\pi f t} \, dt \\
&= \int_{0}^{\infty} \hat{K}(t) e^{-i2\pi f t} \, dt - \int_{0}^{\infty} \hat{K}(t) e^{i2\pi f t} \, dt \\
&= 2i \, \text{Imag} \int_{0}^{\infty} \hat{K}(t) e^{-i2\pi f t} \, dt \\
&= 2i \, \text{Imag} \int_{0}^{\infty} K(t) e^{-i2\pi f t} \, dt - 2i K_0 \, \text{Imag} \int_{0}^{\infty} e^{-\alpha t} e^{-i2\pi f t} \, dt \\
&= 2i \, \text{Imag} \, R(f) + i \frac{4\pi K_0 f}{\alpha^2 + (2\pi f)^2}.
\end{aligned}
\tag{4.24}
$$

Thus, given $R(f)$, we can compute $\hat{R}(f)$ from equation (4.24). Taking the inverse transform of $\hat{R}(f)$ gives us $\hat{K}(t)$. We can then calculate $K(t)$ from equation (4.23).

The application of this technique to a problem in acoustics is illustrated in figure 4.1 [12]. Here we have the pressure response due to an impulse acceleration of a radiating surface element. Notice that the response is zero for negative time, jumps to a finite value at time zero, and then decays to zero at large times. Figure 4.2 shows the same time response computed by numerically performing an inverse Fourier transform of the frequency response without employing the modifications suggested in this section. Notice that the response is not as smooth and doesn't have the right behavior for small times. Since there is a jump at time zero, the inverse Fourier transform converges to the average of the right and left hand limits (0.5 in this case).
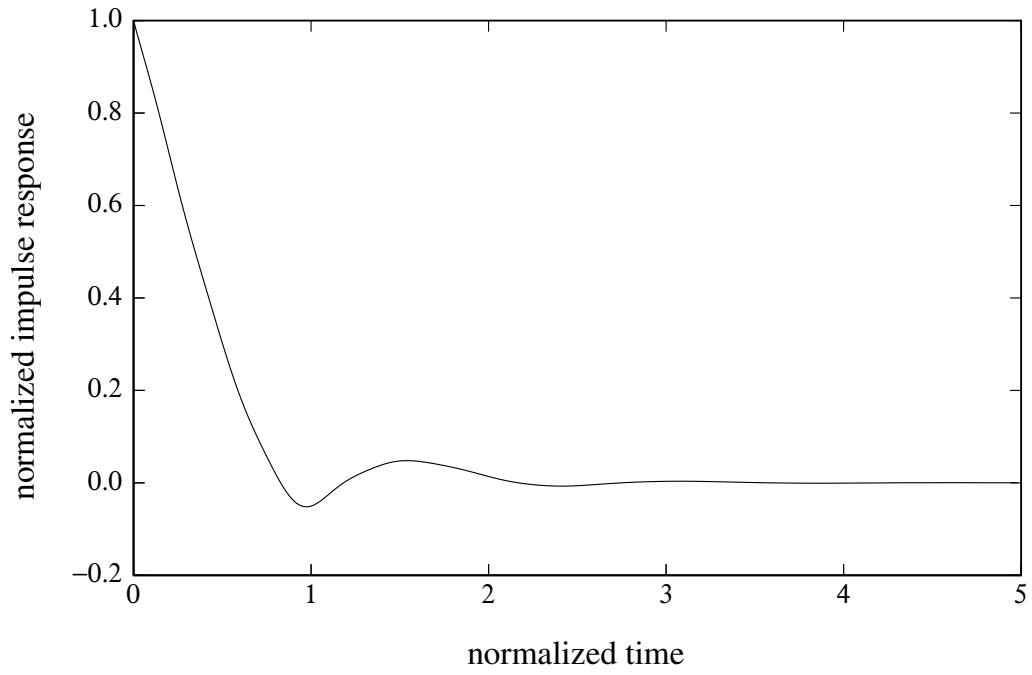
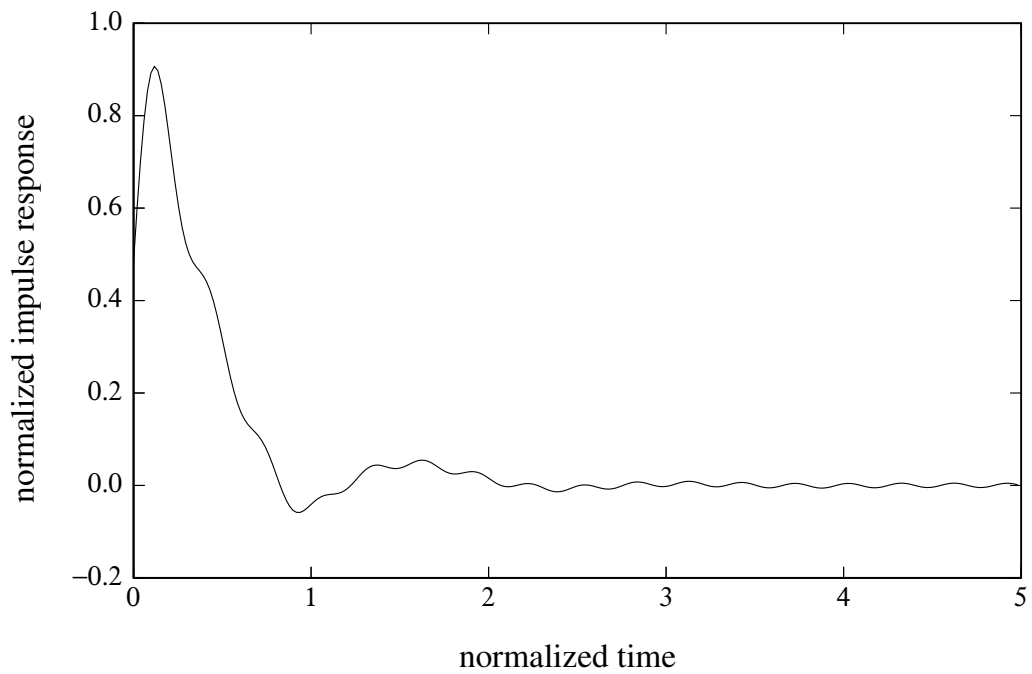Figure 4.1: Pressure impulse response computed by the method described in this section.

Figure 4.2: Pressure impulse response calculated by a direct inversion of the frequency response.

54

5 Discrete Fourier Transform

The Discrete Fourier Transform (DFT) does not involve functions of a continuous variable as is the case with Fourier Series and Fourier Transforms, but only involves a finite set of data points. For this reason it plays a dominant role in fields such as digital signal processing. We will see that the DFT can be interpreted as an approximation to Fourier Series and Fourier Transforms, but it is also a legitimate transform in its own right. It has many properties that are analogous to its continuous counterparts. However, the derivation of these properties is much simpler in the discrete case. We will find some cases, such as the connection of the DFT with finite order rotational symmetry, where the DFT produces an exact answer and not an approximation.

5.1 Interpolation and the Origin of the DFT

As was mentioned in the introduction, the Discrete Fourier Transform arose in planetary orbit interpolation problems. Suppose that a planetary orbit of interest lies in a plane and let us choose an origin such as the earth or the sun. Let θ be an angular variable around the origin and let $r(\theta)$ be the distance from the origin to the planet at the angle θ. Clearly $r(\theta)$ is periodic with period 2π. Suppose we attempt to interpolate $r(\theta)$ by a function $\hat{r}(\theta)$ of the form

$$\hat{r}(\theta) = a_0 + \sum_{n=1}^{N} a_n \cos n\theta + \sum_{n=1}^{N} b_n \cos n\theta, \tag{5.1}$$

i.e., we seek to interpolate $r(\theta)$ by a trigonometric sum. The function $\hat{r}$ can also be written in the exponential form

$$\hat{r}(\theta) = \sum_{m=-N}^{N} c_m e^{im\theta} \tag{5.2}$$

where c_{-m} is the complex conjugate of c_m. Suppose we have $2N + 1$ observations that are equally spaced in angle. We will denote the observation angles by $\theta_0, \ldots, \theta_{2N}$ where

$$\theta_k = \frac{2\pi k}{2N + 1} \qquad k = 0, \ldots, 2N$$

and the observed values by $r_0, \ldots, r_{2N}$. Since $\hat{r}$ is an interpolation formula, it must coincide with the observed values at the observation points, i.e.,

$$\hat{r}(\theta_k) = r(\theta_k) = r_k \qquad k = 0, \ldots, 2N, \tag{5.3}$$

and hence that

$$r_k = \sum_{m=-N}^{N} c_m e^{im\theta_k} = \sum_{m=-N}^{N} c_m e^{i2\pi mk/(2N+1)} \qquad k = 0, \ldots, 2N. \tag{5.4}$$

The complex exponential in equation (5.4) is periodic in m with period $2N + 1$. Let us extend the sequence $c_{-N}, \ldots, c_N$ to a periodic sequence with period $2N + 1$. Then we can allow m in equation (5.4) to range over any set of $2N + 1$ consecutive integers. In particular,

$$r_k = \sum_{m=0}^{2N} c_m e^{i2\pi mk/(2N+1)} \qquad k = 0, \ldots, 2N. \tag{5.5}$$

If we define $M = 2N + 1$, then equation (5.5) can be written

$$r_k = \sum_{m=0}^{M-1} c_m e^{i2\pi mk/M} \qquad k = 0, \ldots, M - 1. \tag{5.6}$$

The coefficients c_m are the solution of the system of linear equations (5.6). We will show that the complex exponentials satisfy a discrete orthogonality condition that simplifies the solution of this system of equations. Let

$$s_m(\theta) = e^{im\theta}. \tag{5.7}$$

Then

$$\sum_{k=0}^{M-1} s_m(\theta_k)\overline{s_n(\theta_k)} = \sum_{k=0}^{M-1} e^{i2\pi(m-n)k/M} = \frac{1 - e^{i2\pi(m-n)}}{1 - e^{i2\pi(m-n)/M}} = 0 \qquad m \neq n \tag{5.8a}$$

$$= M \qquad m = n \tag{5.8b}$$

where the overbar denotes complex conjugate. Here we have used the following expression for the sum of a geometric sequence

$$\sum_{k=0}^{M-1} x^k = \begin{cases} (1 - x^M)/(1 - x) & x \neq 1 \\ M & x = 1. \end{cases} \tag{5.9}$$

This expression for the sum of a geometric series can easily be derived as follows:

1. Let $S = \sum_{k=0}^{M-1} x^k = 1 + x + x^2 + \cdots + x^{M-1}$.

2. Then $xS = x + x^2 + \cdots + x^{M-1} + x^M$.

3. Subtracting xS from S, we get $(1 - x)S = 1 - x^M$ (the other powers of x cancel).

4. As long as $x \neq 1$ we can solve for S and obtain $S = (1 - x^M)/(1 - x)$.

5. Clearly $\sum_{k=0}^{M-1} x^k = M$ when $x = 1$.

The system of equations (5.6) can be written

$$r_k = \sum_{m=0}^{M-1} c_m s_m(\theta_k) \qquad k = 0, \ldots, M - 1. \tag{5.10}$$

Using the orthogonality relations in equation (5.8), we obtain

$$\sum_{k=0}^{M-1} r_k \overline{s_n(\theta_k)} = \sum_{m=0}^{M-1} c_m \sum_{k=0}^{M-1} s_m(\theta_k) \overline{s_n(\theta_k)} = M c_n \tag{5.11}$$

or

$$M c_n = \sum_{k=0}^{M-1} r_k e^{-i2\pi nk/M}. \tag{5.12}$$

Once we have computed $c_0, \ldots, c_{M-1}$ using equation (5.12), then the values c_{-n} in equation (5.4) can be obtained using the periodicity of the sequence c_n. In view of equation (5.6), we have

$$r_k = \frac{1}{M} \sum_{n=0}^{M-1} M c_n e^{i2\pi nk/M}. \tag{5.13}$$

In equation (5.12) we say that $M c_0, \ldots, M c_{M-1}$ is the Discrete Fourier Transform (DFT) of $r_0, \ldots, r_{M-1}$. Likewise, in equation (5.13) we say that $r_0, \ldots, r_{M-1}$ is the Inverse Discrete Fourier Transform of $M c_0, \ldots, M c_{M-1}$. In general, the DFT of a sequence $x_0, \ldots, x_{N-1}$ is written

$$X_n = \sum_{m=0}^{N-1} x_m e^{-i2\pi mn/N} \qquad n = 0, \ldots, N - 1 \tag{5.14}$$

and the Inverse DFT is written

$$x_m = \frac{1}{N} \sum_{n=0}^{N-1} X_n e^{i2\pi mn/N} \qquad m = 0, \ldots, N - 1. \tag{5.15}$$

Equations (5.14) and (5.15) can be written in the matrix form

$$X = Ex$$
$$x = \overline{E}X/N$$

where E is the matrix with components

$$E_{mn} = e^{-i2\pi mn/N} \qquad m, n = 0, \ldots, N-1,$$

X is a column vector with components X_n, and x is a column vector with components x_n. Clearly $\overline{E}/N$ is the inverse of E. Since the complex exponentials $e^{\pm i2\pi mn/N}$ are periodic with period N, the DFT has a natural periodic extension of period N, i.e., $X_{n+kN} = X_n$. In this way we see that X_{N-k} corresponds to X_{-k}.

We have looked at interpolation in terms of sums of complex exponentials. In practice, the function to be interpolated is often an even function which leads to an interpolation function involving only cosines. In particular, we will see later on that interpolation by Chebyshev polynomials is based on cosine series interpolation. We can define odd functions in the discrete case as those sequences in which $x_{-k} = x_{N-k} = x_k$. Suppose we are given samples $x_0, \ldots, x_N$ of a function x. We can extend this sequence to an even sequence of length $2N$ as follows

$$x_{2N-k} = x_k \qquad \text{for } k = 1, \ldots, N-1. \tag{5.16}$$

The DFT of this extended sequence is given by

$$X_n = \sum_{m=0}^{2N-1} x_m e^{-i2\pi mn/2N} = \sum_{m=0}^{2N-1} x_m e^{-i\pi mn/N} \qquad n = 0, \ldots, 2N-1. \tag{5.17}$$

Using the relation in equation (5.16), it follows that

$$X_n = \sum_{m=0}^{N} x_m e^{-i\pi mn/N} + \sum_{m=N+1}^{2N-1} x_m e^{-i\pi mn/N}$$

$$= \sum_{m=0}^{N} x_m e^{-i\pi mn/N} + \sum_{m=N-1}^{1} x_{2N-m} e^{-i\pi(2N-m)n/N}$$

$$= \sum_{m=0}^{N} x_m e^{-i\pi mn/N} + \sum_{m=1}^{N-1} x_m e^{+i\pi mn/N}$$

$$= \left[x_0 + (-1)^n x_N\right] + 2\sum_{m=1}^{N-1} x_m \cos(\pi mn/N). \tag{5.18}$$

We define the Discrete Cosine Transform (DCT) $\hat{X}_0, \ldots, \hat{X}_{N-1}$ by

$$\hat{X}_n = \tfrac{1}{2}[x_0 + (-1)^n x_N] + \sum_{m=1}^{N-1} x_m \cos(\pi mn/N) \qquad n = 0, \ldots, 2N-1. \tag{5.19}$$

Clearly, $\hat{X}_n = X_n/2$. It follows from equation (5.18) that the sequence X_n is also even, i.e., $X_{2N-n} = X_n$. Taking the inverse DFT of the sequence X_n, we obtain

$$x_m = \frac{1}{2N} \sum_{n=0}^{2N-1} X_n e^{i\pi mn/N} \qquad m = 0, \ldots, 2N-1. \tag{5.20}$$

Since X_n is an even sequence it follows that

$$x_m = \frac{1}{2N}\left[X_0 + (-1)^m X_N\right] + \frac{1}{N}\sum_{n=1}^{N-1} X_n \cos(\pi mn/N)$$

$$= \frac{2}{N}\left\{\tfrac{1}{2}[\hat{X}_0 + (-1)^m \hat{X}_N] + \sum_{n=1}^{N-1} \hat{X}_n \cos(\pi mn/N)\right\} \qquad m = 0, \ldots, 2N-1. \tag{5.21}$$

We take equation (5.21) as the definition of the Inverse Discrete Cosine Transform. Notice that the inverse DCT has the same form as the DFT except for the for the factor $2/N$.

Getting back to the interpolation problem, the above results suggest that we choose an interpolation function of the form

$$x(\theta) = \sum_{n=0}^{N} c_n \cos n\theta. \tag{5.22}$$

Since $x(\theta)$ must coincide with x_m at $\theta = \theta_m = \pi m / N$, we have

$$x_m = \sum_{n=0}^{N} c_n \cos(\pi m n / N) \qquad m = 0, \dots, N. \tag{5.23}$$

It follows from equations (5.19), (5.21) and (5.23) that

$$c_0 = \frac{1}{N} \hat{X}_0 = \frac{1}{N} \left[\tfrac{1}{2}(x_0 + x_N) + \sum_{m=1}^{N-1} x_m \right] \tag{5.24a}$$

$$c_N = \frac{1}{N} \hat{X}_N = \frac{1}{N} \left\{ \tfrac{1}{2}\left[x_0 + (-1)^N x_N\right] + \sum_{m=1}^{N-1} (-1)^m x_m \right\} \tag{5.24b}$$

$$c_n = \frac{2}{N} \hat{X}_n = \frac{2}{N} \left\{ \tfrac{1}{2}\left[x_0 + (-1)^n x_N\right] + \sum_{m=1}^{N-1} x_m \cos(\pi m n / N) \right\} \qquad n = 1, \dots, N-1.$$

$$\tag{5.24c}$$

Thus the coefficients in the interpolation formula (5.22) can be obtained using the Discrete Cosine Transform. There are fast algorithms for computing the DFT that are called FFT (Fast Fourier Transform) algorithms. Similarly, there are fast algorithms for computing the DCT that make use of FFT algorithms.

5.2 Properties of the DFT

In this section we will look at some of the properties of the Discrete Fourier Transform. Many of the properties have analogs in the continuous case, but some of the properties are unique to the discrete case. Since the complex exponentials $e^{\pm i2\pi mn/N}$ in equations (5.14) and (5.15) are periodic with period N, the sequence $x_0, \ldots, x_{N-1}$ and its transform $X_0, \ldots, X_{N-1}$ can be extended in a natural way to periodic sequences of period N. In the properties that follow, it will be assumed that the sequences involved have been extended to periodic sequences of period N. In the definitions of the DFT and its inverse, the summations run from 0 to $N-1$. However, because of the periodicity, the same result would be obtained if the index was allowed to range over any N consecutive integers.

Units The exponential terms that appear in the definitions of the DFT and its inverse are unitless. Sometimes it is useful to introduce units into the discussion. A common case is to identify m and n with time and frequency. Let the times $t_0, \ldots, t_{N-1}$ be defined by $t_m = m\Delta t$, $m = 0, \ldots, N-1$, where Δt is a specified time increment. To keep the exponential terms dimensionless we define frequencies $f_0, \ldots, f_{N-1}$ by $f_n = n\Delta f$, $n = 0, \ldots, N-1$, where $\Delta f = 1/(N\Delta t)$. Then

$$e^{\pm i2\pi mn/N} = e^{\pm i2\pi(m\Delta t)n/(N\Delta t)} = e^{\pm i2\pi f_n t_m}.$$

The frequency increment is the reciprocal of the total time span $N\Delta t$. Similarly, the time increment is the reciprocal of the total frequency span $N\Delta f$. In other problems different units might be appropriate. They can be introduced in the same way.

Real Sequences Suppose $x_0, \ldots, x_{N-1}$ is a real sequence. Then the DFT X of x has the property

$$X_{-n} = X_{N-n} = \sum_{m=0}^{N-1} x_m e^{-i2\pi m(-n)/N} = \sum_{m=0}^{N-1} x_m e^{i2\pi mn/N} = \overline{X_n}. \qquad (5.25)$$

That is, the components of X for plus and minus n are complex conjugates of each other.

Even and Odd Sequences Suppose x is an even sequence, i.e., $x_{-n} = x_n$ for all n. Then the DFT of x has the property

$$X_{-n} = \sum_{m=0}^{N-1} x_m e^{i2\pi mn/N}$$

$$= \sum_{m=-(N-1)}^{0} x_m e^{i2\pi mn/N}$$

$$= \sum_{m=0}^{N-1} x_{-m} e^{-i2\pi mn/N}$$

$$= \sum_{m=0}^{N-1} x_m e^{-i2\pi mn/N} = X_n,$$

i.e., the DFT of an even sequence is also an even sequence. Applying the same type of argument to odd sequences ($x_{-n} = -x_n$), it can be shown that the DFT of an odd sequence is also odd. Combining these results with the previous result for real sequences, we see that the DFT of a real and even sequence is real and even, and the DFT of a real and odd sequence is imaginary and odd.

Shift Theorem Suppose we have a sequence of values $x_0, \ldots, x_{N-1}$. Consider the sequence y_m defined by

$$y_m = x_{m-k} \qquad m = 0, \ldots, N-1 \tag{5.26}$$

for some fixed integer k. Taking the DFT of the y_m sequence, we get

$$Y_n = \sum_{m=0}^{N-1} y_m e^{-i2\pi mn/N} = \sum_{m=0}^{N-1} x_{m-k} e^{-i2\pi mn/N}$$

$$= \sum_{m=-k}^{N-1-k} x_m e^{-i2\pi(m+k)n/N} = e^{-i2\pi kn/N} \sum_{m=0}^{N-1} x_m e^{-i2\pi mn/N} = e^{-i2\pi kn/N} X_n. \tag{5.27}$$

Thus, shifting the index m in the original x_m sequence by k results in multiplying the transform component X_n by the exponential factor $e^{-i2\pi kn/N}$. This result is called the Shift Theorem.

Convolution Theorem If $x_0, \ldots, x_{N-1}$ and $y_0, \ldots, y_{N-1}$ are two sequences of length N, then we define their convolution $x \circledast y$ by

$$(x \circledast y)_n = \sum_{m=0}^{N-1} x_m y_{n-m}. \tag{5.28}$$

Because of the periodicity of x and y, this convolution is sometimes called *circular convolution* or *cyclic convolution*. Making a change of index in equation (5.28), we see that

$$(x \circledast y)_n = \sum_{m=0}^{N-1} x_m y_{n-m} = \sum_{k=n-N+1}^{n} x_{n-k} y_k = \sum_{k=0}^{N-1} x_{n-k} y_k = (y \circledast x)_n, \qquad (5.29)$$

i.e., $x \circledast y = y \circledast x$. Let us now look at the DFT of $x \circledast y$. Let

$$z_m = (x \circledast y)_m = \sum_{k=0}^{N-1} x_k y_{m-k}. \qquad (5.30)$$

Then, the DFT of z is given by

$$
\begin{aligned}
Z_n &= \sum_{m=0}^{N-1} z_m e^{-i2\pi mn/N} \\
&= \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} x_k y_{m-k} e^{-i2\pi mn/N} \\
&= \sum_{k=0}^{N-1} x_k \sum_{m=0}^{N-1} y_{m-k} e^{-i2\pi mn/N} \\
&= \sum_{k=0}^{N-1} x_k e^{-i2\pi kn/N} Y_n \qquad \text{(by the shift theorem)} \\
&= X_n Y_n. \qquad (5.31)
\end{aligned}
$$

Thus, the DFT of $x \circledast y$ has the components $X_n Y_n$, $n = 0, \ldots, N-1$. This result is called the Convolution Theorem. We can use this theorem to construct an alternate method for evaluating circular convolutions. We can take the DFT of the sequences x and y, multiply corresponding components of X and Y, and then take the inverse DFT of the result. This method works fine when dealing with periodic functions, but in practice this is seldom the case. Let x and y be infinite sequences such that $x_n = 0$ for $n < 0$ and $n > N$, and $y_n = 0$ for $n < 0$ and $N > M$. We define a non cyclic convolution $x * y$ by

$$(x * y)_n = \sum_{m=-\infty}^{\infty} x_m y_{n-m}.$$

Although we have allowed the index in the summation to go from $-\infty$ to ∞, only a finite number of the terms are nonzero. Moreover, $(x * y)_n$ is only nonzero for $n = 0, \ldots, M+N$. This type of

63

convolution arises in a number of circumstances. In signal processing it arises in connection with time-limited or approximately time-limited sequences. Many digital filters can be represented by convolutions of this type in which one of the sequences represents a sampled signal and the other represents the impulse response of the filter. Another interesting application concerns the multiplication of two polynomials. Let $p(x)$ and $q(x)$ be the two polynomials

$$p(x) = a_0 + a_1 x + \cdots + a_M x^M$$
$$q(x) = b_0 + b_1 x + \cdots + b_N x^N \tag{5.32}$$

and let $r(x)$ be their product. $r(x)$ can be written in the form

$$r(x) = c_0 + c_1 x + \cdots + c_{M+N} x^{M+N}. \tag{5.33}$$

The coefficients c_n of $r(x)$ are given by

$$c_n = (a * b)_n = \sum_{m=-\infty}^{\infty} a_m b_{n-m} \tag{5.34}$$

where we have defined a_n to be zero outside the range $0 \le n \le M$ and b_n to be zero outside the range $0 \le n \le N$.

It turns out that we can evaluate noncyclic convolutions using cyclic convolutions if we add an appropriate number of zeros to the sequences. Let x and y be infinite sequences defined as before. Define sequences $\hat{x}$ and $\hat{y}$ of length $N + M + 1$ by

$$\hat{x}_n = \begin{cases} x_n & 0 \le n \le N \\ 0 & N + 1 \le n \le M + N \end{cases} \tag{5.35}$$

$$\hat{y}_n = \begin{cases} y_n & 0 \le n \le M \\ 0 & M + 1 \le n \le M + N. \end{cases} \tag{5.36}$$

$$\tag{5.37}$$

Then

$$(\hat{x} \circledast \hat{y})_n = \sum_{m=0}^{N+M} \hat{x}_m \hat{y}_{n-m} = \sum_{m=0}^{N} x_m \hat{y}_{n-m} \qquad 0 \le n \le M + N. \tag{5.38}$$

Suppose $n - m < 0$. Then by periodicity

$$\hat{y}_{n-m} = \hat{y}_{N+M+1+n-m}.$$

Looking at the index $N + M + 1 + n - m$, we see that $N + M + 1 + n - m \geq N + M + 1 + 0 - N = M + 1$ and $N + M + 1 + n - m \leq N + M + 1 + (-1) = N + M$. It follows from the definition of $\hat{y}$ in equation (5.36) that $\hat{y}_{n-m} = 0$ for $n - m < 0$. It also follows from the definition of $\hat{y}$ that $\hat{y}_{n-m} = y_{n-m}$ when $0 \leq n - m \leq M$. The index $n - m$ is always less than $M + N$. If $n - m > M$, then it again follows from the definition of $\hat{y}$ that $\hat{y}_{n-m} = 0$. Thus, equation (5.38) can be written

$$(\hat{x} \circledast \hat{y})_n = \sum_{m=0}^{N} x_m y_{n-m} = \sum_{m=-\infty}^{\infty} x_m y_{n-m} = (x * y)_n. \tag{5.39}$$

We have shown that the nonzero values of $x * y$ can be obtained by cyclic convolution.

Discrete Delta Function   The discrete analog of the delta function is defined by

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases} \tag{5.40}$$

Clearly, the discrete delta function δ has the property

$$(\delta * x)_n = \sum_{m=0}^{N-1} \delta_m x_{n-m} = x_n, \tag{5.41}$$

i.e., $\delta * x = x$. The DFT D of δ is given by

$$D_n = \sum_{m=0}^{N-1} \delta_m e^{-i2\pi mn/N} = 1 \qquad n = 0, \ldots, N - 1. \tag{5.42}$$

Parseval's Theorem   Let $x_0, \ldots, x_{N-1}$ be a complex sequence and define a sequence $\hat{x}$ by

$$\hat{x}_m = \overline{x_{-m}} \qquad m = 0, \ldots, N - 1.$$

The DFT of $\hat{x}$ is given by

$$\hat{X}_n = \sum_{m=0}^{N-1} \hat{x}_m e^{i2\pi mn/N}$$

$$= \sum_{m=0}^{N-1} \hat{x}_{-m} e^{-i2\pi mn/N} \qquad m \to -m$$

$$= \sum_{m=0}^{N-1} \overline{x_m} e^{-i2\pi mn/N}$$

$$= \overline{\sum_{m=0}^{N-1} x_m e^{i2\pi mn/N}}$$

$$= \overline{X_n} \qquad n = 0, \ldots, N-1.$$

It follows from the definition of the cyclic convolution that

$$(x \circledast \hat{x})_0 = \sum_{m=0}^{N-1} x_m \hat{x}_{-m} = \sum_{m=0}^{N-1} x_m \overline{x_m} = \sum_{m=0}^{N-1} |x_m|^2. \tag{5.43}$$

By the convolution theorem and the definition of the inverse DFT, we have

$$(x \circledast \hat{x})_m = \frac{1}{N} \sum_{n=0}^{N-1} X_n \overline{X_n} e^{i2\pi mn/N}.$$

Therefore,

$$(x \circledast \hat{x})_0 = \frac{1}{N} \sum_{n=0}^{N-1} X_n \overline{X_n} = \frac{1}{N} \sum_{n=0}^{N-1} |X_n|^2. \tag{5.44}$$

Combining equations (5.43) and (5.44), we get

$$\sum_{m=0}^{N-1} |x_m|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X_n|^2.$$

This result is known as Parseval's Theorem or sometimes as Rayleigh's Energy Theorem.

Downsampling and Aliasing Suppose we have a sequence $x_0, \ldots, x_{N-1}$ where $N = ML$. Downsampling by L produces a sequence $y_0, \ldots, y_{M-1}$ where $y_m = x_{mL}$, i.e., y consists of

every L-th sample of x. It follows from the sum formula for a geometric series in equation (5.9) that

$$\sum_{l=0}^{L-1} e^{-i2\pi ln/L} = \frac{1-e^{-i2\pi n}}{1-e^{-i2\pi n/L}} = \begin{cases} L & n = 0(\mod L) \\ 0 & n \neq 0(\mod L). \end{cases}$$

Thus, the DFT of the downsampled sequence $y_0, \ldots, y_{M-1}$ is given by

$$\begin{aligned}
Y_k &= \sum_{m=0}^{M-1} y_m e^{-i2\pi km/M} \\
&= \sum_{m=0}^{M-1} x_{mL} e^{-i2\pi km/M} \\
&= \frac{1}{L} \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N} \sum_{l=0}^{L-1} e^{-i2\pi ln/L} \\
&= \frac{1}{L} \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N} \sum_{l=0}^{L-1} e^{-i2\pi lMn/N} \\
&= \frac{1}{L} \sum_{l=0}^{L-1} \sum_{n=0}^{N-1} x_n e^{-i2\pi (k+lM)n/N} \\
&= \frac{1}{L} \sum_{l=0}^{L-1} X_{k+lM} \qquad k = 0, \ldots, M-1.
\end{aligned} \tag{5.45}$$

The sequence $Y_0, \ldots, Y_{M-1}$ is called an aliased version of X. Thus, downsampling the original sequence results in aliasing of its discrete transform.

Stretching and Repeating Let $N = ML$. The sequence $x_0, \ldots, x_{M-1}$ is said to be stretched by L to the sequence $y_0, \ldots, y_{N-1}$ if

$$y_m = \begin{cases} x_{m/L} & m/L \text{ an integer} \\ 0 & m/L \text{ not an integer.} \end{cases} \tag{5.46}$$

Thus stretching by L places $L-1$ zeroes between each sample of x. The DFT of the sequence y is given by

$$Y_n = \sum_{m=0}^{N-1} y_m e^{-i2\pi mn/N}$$

$$= \sum_{k=0}^{M-1} x_k e^{-i2\pi(kL)n/N}$$

$$= \sum_{k=0}^{M-1} x_k e^{-i2\pi kn/M} \qquad n = 0, \dots, N-1 \qquad (5.47)$$

If X is the DFT of x, then Y consists of the sequence $X_0, \dots, X_{M-1}$ repeated L times.

Zero-Padding and Interpolation Let $x_0, \dots, x_{N-1}$ be a real sequence. Then the transform sequence $X_0, \dots, X_{N-1}$ has the property $X_{-k} = X_{N-k} = \overline{X_k}$. In particular, for N even, $N/2$ is real. Because of the periodicity of X, we can allow the index to range over any N consecutive integers. For zero-padding it is convenient to center the transform with the $n = 0$ value in the center. For N odd this is easy to do. For example, in place of the sequence X_0, X_1, X_2, X_3, X_4, we could consider the sequence $X_{-2}, X_{-1}, X_0, X_1, X_2$ or equivalently X_3, X_4, X_0, X_1, X_2. Thus, for N odd, we could write

$$x_m = \frac{1}{N} \sum_{n=0}^{N-1} X_n e^{i2\pi mn/N} = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} X_n e^{i2\pi mn/N}. \qquad (5.48)$$

Consider the real-valued function

$$x(\theta) = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} X_n e^{in\theta}. \qquad (5.49)$$

This function has the property that $x(\theta_m) = x_m$ for $\theta_m = 2\pi m/N$. That is the sequence x represent sampled values of the function $x(\theta)$ at the equally space points θ_m. If we had not centered the X values prior to replacing $2\pi m/N$ by θ, the resulting x function would still coincide with x_m at θ_m, but it would not be real-value for all θ. Suppose we now expand the sequence $X_{-(N-1)/2}, \dots, X_{(N-1)/2}$ to M terms ($M > N$) by adding zeroes to the two ends. We refer to this expanded sequence as a zero-padded sequence. It does not matter how many of the zero values we place at each end, for when we use periodicity to change the indices back to the range $0, \dots, M-1$, the zeroes will all be consecutive. Let us denote by $\hat{x}_m$, $m = 0, \dots, M-1$, the values obtained by taking the inverse DFT of the zero-padded sequence. Then

$$\hat{x}_m = \frac{1}{M} \sum_{n=-(N-1)/2}^{(N-1)/2} X_n e^{i2\pi mn/M} \qquad m = 0, \dots, M-1. \qquad (5.50)$$

It follows from equation (5.50) that $\hat{x}_m = (N/M)x(\hat{\theta}_m)$ where $\hat{\theta}_m = 2\pi m/M$ for $m = 0, \ldots, M-1$. Thus, apart from the factor N/M, the sequence $\hat{x}$ also represents sampled values of the function $x(\theta)$, but on a finer grid of points. If M is an integer multiple of N, say $M = LN$, then $\hat{\theta}_{mL} = \theta_m$ for all m and we have effectively interpolated between the values of the original sequence. To take the inverse DFT we need to convert the index range to $0, \ldots, M-1$. To do this we can first set all the components of the zero-padded sequence $\hat{X}_0, \ldots, \hat{X}_{M-1}$ to zero. Next we set $\hat{X}_m = (M/N)X_m$ for $m = 0, \ldots, (N-1)/2$. Finally, we set $\hat{X}_{M-m} = (M/N)X_{N-m}$ for $m = 1, \ldots, (N-1)/2$. The inverse DFT of this sequence will give sampled values of $x(\theta)$.

The case for even N requires a little more care since we can't directly center values around $n = 0$ with the same number on each side. One approach is to neglect the $N/2$ term and work with

$$x_m \doteq \frac{1}{N} \sum_{n=-(N/2-1)}^{N/2-1} X_n e^{-i2\pi mn/N} \qquad m = 0, \ldots, N-1. \qquad (5.51)$$

You can then proceed as in the odd case. Another way of handling the even n case is to split the $N/2$ value between the indices $\pm N/2$, i.e., we work with

$$x_m = \frac{1}{N} \sum_{n=-N/2}^{N/2} Y_n e^{-i2\pi mn/N} \qquad m = 0, \ldots, N-1 \qquad (5.52)$$

where $Y_n = X_n$ for $n \neq \pm N/2$ and $Y_{N/2} = Y_{-N/2} = \frac{1}{2}X_{N/2}$. This gives the correct value for x_m since $X_{N/2}$ is real and

$$e^{-i2\pi(N/2)n/N} = e^{i2\pi(N/2)n/N}.$$

In this case we zero pad as follows:

1. Set $\hat{X}_m = 0$ for $m = 0, \ldots, M-1$

2. Set $\hat{X}_m = (M/N)X_m$ for $m = 0, \ldots, N/2-1$

3. Set $\hat{X}_{M-m} = (M/N)X_{N-m}$ for $m = 1, \ldots, N/2-1$

4. Set $\hat{X}_{N/2} = \hat{X}_{M-N/2} = \frac{1}{2}(M/N)X_{N/2}$.

The inverse DFT of this sequence gives sampled values of the real-valued function

$$x(\theta) = \frac{1}{N} \sum_{n=-N/2}^{N/2} Y_n e^{-in\theta}. \qquad (5.53)$$

69

There are many more properties of the DFT that could have been included, but the ones given here are some of the more important ones. One of the nice things about Discrete Fourier Transforms is that the derivation of its properties does not generally involve complicated mathematics.

5.3 Relation between the Discrete Fourier Transform and Fourier Series

Suppose $x(t)$ is periodic with period T. Then x has a Fourier series expansion

$$x(t) = \sum_{n=-\infty}^{\infty} c(n)e^{i2\pi nt/T} \tag{5.54}$$

with

$$c(n) = \frac{1}{T} \int_0^T x(t)e^{-i2\pi nt/T}\,dt. \tag{5.55}$$

Let us sample x at N equally spaced points in $[0, T)$. If $\Delta t = T/N$, then

$$
\begin{aligned}
x(m\Delta t) &= \sum_{n=-\infty}^{\infty} c(n)e^{i2\pi mn/N} \\
&= \sum_{k=-\infty}^{\infty} \sum_{n=kN}^{(k+1)N-1} c(n)e^{i2\pi mn/N} \\
&= \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} c(n+kN)e^{i2\pi m(n+kN)/N} \\
&= \sum_{n=0}^{N-1} \sum_{k=-\infty}^{\infty} c(n+kN)e^{i2\pi mn/N} \\
&= \sum_{n=0}^{N-1} c_p(n)e^{i2\pi mn/N} \qquad m = 0, 1, \ldots, N-1
\end{aligned}
\tag{5.56}
$$

where

$$c_p(n) = \sum_{k=-\infty}^{\infty} c(n+kN). \tag{5.57}$$

Here we have used the fact that $e^{i2\pi mn/N}$ is periodic with period N. It follows from equation (5.56) that $\{x(m\Delta t)\}$ and $\{Nc_p(n)\}$ are a discrete Fourier transform pair. Moreover,

$$c_p(N-n) = \sum_{k=-\infty}^{\infty} c(-n+kN). \tag{5.58}$$

71

Taking the inverse DFT of equation (5.56), we obtain

$$c_p(n) = \frac{1}{N} \sum_{m=0}^{N-1} x(m\Delta t)e^{-i2\pi mn/N} \qquad n = 0, 1, \ldots, N - 1. \qquad (5.59)$$

Suppose N is chosen so that $c(n) \doteq 0$ for $|n| \geq N/2$. Then it follows from equations (5.57) and (5.58) that

$$c_p(n) \doteq c(n)$$

and

$$c_p(N - n) \doteq c(-n) \qquad \text{for } n = 0, 1, \ldots, N/2.$$

Aliasing occurs if terms with $k \neq 0$ contribute significantly to the sums in equations (5.57) and (5.58). To prevent aliasing it is necessary for $c(n)$ to be very small when $|n| \geq N/2$. Notice that the values of $c(n)$ for negative n occur in the last half of $c_p(n)$ in reverse order.

5.4 Approximation of the Fourier Transform by the Discrete Fourier Transform

Let $x(t)$ and $X(f)$ form a Fourier transform pair, i.e.,

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}\, df \tag{5.60}$$

and

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}\, dt. \tag{5.61}$$

If we sample $x(t)$ at intervals of Δt, then

$$x(m\Delta t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi fm\Delta t}\, df \tag{5.62}$$

$$= \int_{-\infty}^{\infty} X(f)e^{i2\pi mf/F}\, df \tag{5.63}$$

$$= \sum_{k=-\infty}^{\infty} \int_{kF}^{(k+1)F} X(f)e^{i2\pi mf/F}\, df \tag{5.64}$$

where $F = 1/\Delta t$. By change of variable in the above integrals

$$\int_{kF}^{(k+1)F} X(f)e^{i2\pi mf/F}\, df = \int_0^F X(f+kF)e^{i2\pi mf/F}\, df. \tag{5.65}$$

Thus, it follows from equations (5.64)–(5.65) that

$$x(m\Delta t) = \int_0^F X_p(f)e^{i2\pi mf/F}\, df \tag{5.66}$$

where

$$X_p(f) = \sum_{k=-\infty}^{\infty} X(f+kF). \tag{5.67}$$

Since $X_p(f)$ is periodic with period F, it can be expanded in a Fourier series, i.e.,

73

$$X_p(f) = \sum_{l=-\infty}^{\infty} a_l e^{-i2\pi lf/F}. \tag{5.68}$$

The coefficients a_l are given by

$$a_l = \frac{1}{F} \int_0^F X_p(f) e^{i2\pi lf/F} \, df. \tag{5.69}$$

It follows from equations (5.66) and (5.69) that

$$a_l = \frac{1}{F} x(l\Delta t) \tag{5.70}$$

and hence

$$X_p(f) = \frac{1}{F} \sum_{l=-\infty}^{\infty} x(l\Delta t) e^{-i2\pi lf/F}. \tag{5.71}$$

Let $T = N\Delta t$ and $\Delta f = 1/T = F/N$. Then it follows from equation (5.71) that

$$\begin{aligned} X_p(n\Delta f) &= \frac{1}{F} \sum_{l=-\infty}^{\infty} x(l\Delta t) e^{-i2\pi ln/N} \\ &= \frac{1}{F} \sum_{k=-\infty}^{\infty} \sum_{m=kN}^{(k+1)N-1} x(m\Delta t) e^{-i2\pi mn/N}. \end{aligned} \tag{5.72}$$

Since

$$\sum_{m=kN}^{(k+1)N-1} x(m\Delta t) e^{-i2\pi mn/N} = \sum_{m=0}^{N-1} x(m\Delta t + kN\Delta t) e^{-i2\pi mn/N}, \tag{5.73}$$

it follows from equation (5.72) that

$$X_p(n\Delta f) = \frac{1}{F} \sum_{m=0}^{N-1} x_p(m\Delta t) e^{-i2\pi mn/N} \tag{5.74}$$

where

74

$$x_p(m\Delta t) = \sum_{k=-\infty}^{\infty} x(m\Delta t + kN\Delta t). \tag{5.75}$$

Taking the discrete inverse Fourier transform of equation (5.74), we get

$$x_p(m\Delta t) = \frac{F}{N} \sum_{n=0}^{N-1} X_p(n\Delta f)e^{i2\pi mn/N} \tag{5.76}$$

or

$$x_p(m\Delta t) = \frac{1}{T} \sum_{n=0}^{N-1} X_p(n\Delta f)e^{i2\pi mn/N}. \tag{5.77}$$

Thus, $\{x_p(m\Delta t)\}$ and $\{FX_p(n\Delta f)\}$ are a discrete Fourier transform pair.

Suppose $X(f)$ is negligible for $|f| \geq F/2$. Then it follows from equation (5.67) that

$$X_p(f) \doteq X(f) \qquad \text{for } 0 \leq f \leq F/2 \tag{5.78}$$

and

$$X_p(F - f) \doteq X(-f) \qquad \text{for } 0 \leq f \leq F/2. \tag{5.79}$$

Thus

$$X_p(n\Delta f) \doteq X(n\Delta f) \qquad \text{for } 0 \leq n \leq N/2 \tag{5.80}$$

and

$$X_p\big((N-n)\Delta f\big) \doteq X(-n\Delta f) \qquad \text{for } 0 \leq n \leq N/2. \tag{5.81}$$

Similarly, if $x(t)$ is negligible for $|t| \geq T/2$, then

$$x_p(m\Delta t) \doteq x(m\Delta t) \qquad \text{for } 0 \leq m \leq N/2 \tag{5.82}$$

and

$$x_p\big((N-m)\Delta t\big) \doteq x(-m\Delta t) \qquad \text{for } 0 \leq m \leq N/2. \tag{5.83}$$

75

Aliasing occurs when terms with $k \neq 0$ contribute significantly to the sum on the right-hand-side of equation (5.67) or (5.75). In order to prevent aliasing it is necessary for $X(f)$ to be negligible when $|f| \geq F/2$. Similarly, it is necessary for $x(t)$ to be negligible when $|t| \geq T/2$. Notice that the values for negative frequencies occur in reverse order in the last half of the sequence $\{X_p(n\Delta f)\}_{n=0}^{N-1}$. Similarly, the values for negative times occur in reverse order in the last half of the sequence $\{x_p(m\Delta t)\}_{m=0}^{N-1}$. If the function $x(t)$ is real-valued, then the negative frequency values are the complex conjugate of the corresponding positive frequency values. Notice also that the frequency values $X_p(n\Delta f)$ must be multiplied by F ($F = 1/\Delta t$) prior to performing the inverse discrete Fourier transform to get the time values $x_p(m\Delta t)$. We can get interpolated values in the time domain by padding the frequency sequence with a sequence of zeroes. These zeroes go in the middle—between the positive and negative frequency values—since this is where the large positive and negative frequency values reside.

5.5 Relation Between the DFT and Finite Order Rotational Symmetry

We say that a geometric body has N-th order rotational symmetry about some axis if a rotation of the body through an angle of $2\pi/N$ looks the same as the original. For example, an octagon has 8-th order rotational symmetry. A body with N-th order rotational symmetry can be subdivided into N identical sub blocks of angular width $2\pi/N$. Suppose that we have a finite number of nodal points within the body that are arranged symmetrically in the N sub blocks. We will number these nodal points in the same order within each block. Suppose we wish to determine the value of some physical quantity at each nodal point. If x is a vector of the nodal point values, then it can be partitioned as follows

$$
x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}
\tag{5.84}
$$

where x_n contains the values in the n-th sub block. Suppose also that x is the solution of a linear set of equations

$$
Ax = b.
\tag{5.85}
$$

We can think of b as some kind of forcing function and x as the resultant. Systems of linear equations like this occur quite often in the solution of physical problems.

We can define a discrete rotation matrix operator R by

$$
R \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{N-1} \\ x_0 \end{pmatrix}.
\tag{5.86}
$$

It is easily seen that the matrix R is given by

$$R = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ I & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{5.87}$$

where I is an identity matrix whose row and column dimension is the number of nodal points within a symmetry block. The matrix R has the property $R^N = I$ where I here is an identity matrix with row and column dimension equal to the total number of nodes. Suppose e is an eigenvector of R with eigenvalue λ, i.e., $Re = \lambda e$. Then

$$R^N e = \lambda^N e = e \tag{5.88}$$

and hence $\lambda^N = 1$, i.e., the eigenvalues of R are the N-th roots of unity. The eigenvalues λ_n of R can be written as follows

$$\lambda_n = e^{i2\pi n/N} \qquad n = 0, 1, \ldots, N - 1. \tag{5.89}$$

Let E be a matrix whose columns are independent eigenvectors of R. One choice for E is the $N \times N$ block matrix in which the block E_{mn} in the m-th block row and n-th block column is given by

$$E_{mn} = e^{i2\pi mn/N} I \qquad m, n = 0, 1, \ldots, N - 1. \tag{5.90}$$

The inverse of the matrix E is $\overline{E}/N$, where the bar over E indicates complex conjugate.

If the body and any relevant physical properties have N-th order rotational symmetry, then a rotation of the force vector b will cause a corresponding rotation of the resultant x, i.e.,

$$ARx = Rb. \tag{5.91}$$

Combining equations (5.85) and (5.91), we get

$$ARx = RAx. \tag{5.92}$$

Since equation (5.92) holds for arbitrary b and hence for arbitrary x, we must have

$$AR = RA. \tag{5.93}$$

Thus the matrix A describing the physical problem must commute with the rotation matrix R. This commutation relation implies that A has the block circulant form

$$A = \begin{pmatrix} A_0 & A_1 & \cdots & A_{N-1} \\ A_{N-1} & A_0 & \cdots & A_{N-2} \\ \vdots & & & \vdots \\ A_1 & \cdots & A_{N-1} & A_0 \end{pmatrix}. \tag{5.94}$$

This can be easily seen by multiplying out the blocks of AR and RA.

If e is an eigenvector of R with eigenvalue λ, then it follows that

$$ARe = RAe = \lambda Ae. \tag{5.95}$$

Thus, Ae is also an eigenvector of R corresponding to the eigenvalue λ, i.e., the eigenspaces of R are invariant under A. This invariance of the eigenspaces of R under A implies that

$$AE = ED \tag{5.96}$$

where D is a block diagonal matrix. The columns of E are independent and thus can be used as a basis. The expansion of x in terms of this basis can be written as follows

$$x = \frac{1}{N}EX \tag{5.97}$$

where $\frac{1}{N}X$ is the vector of expansion coefficients. We can write equation (5.97) in terms of block components as follows

$$x_m = \frac{1}{N} \sum_{n=0}^{N-1} X_n e^{i2\pi mn/N} \qquad m = 0, 1, \ldots, N-1. \tag{5.98}$$

In other words, the vectors x_m can be obtained from the vectors X_n by means of the inverse Discrete Fourier Transform. Substituting equation (5.97) into equation (5.85) and making use of equation (5.96), we obtain

$$Ax = \frac{1}{N}AEX = \frac{1}{N}EDX = b. \tag{5.99}$$

This equation can be reduced to

$$DX = NE^{-1}b = \overline{E}b. \tag{5.100}$$

Since D is block diagonal, we have reduced the large system of equations $Ax = b$ to the solution of N smaller sets of equations. Since solution time goes as the cube of the equation size, this reduction results in a large time saving. Notice that the n-th sub block of $\overline{E}b$ is given by

$$(\overline{E}b)_n = \sum_{m=0}^{N-1} e^{-i2\pi mn/N} b_m. \tag{5.101}$$

Here again we have a Discrete Fourier Transform relation. The block diagonal matrix D is related to A by $D = E^{-1}AE$. It can be shown that the m-th diagonal block D_m is given by

$$D_m = \sum_{n=0}^{N-1} e^{i2\pi mn/N} A_n. \tag{5.102}$$

Thus, we have seen that the Discrete Fourier Transform arises naturally when considering problems having finite order rotational symmetry.

5.6 Interpolation Using Chebyshev Polynomials

In a previous section we looked at trigonometric interpolation. The problem with this form of interpolation is that it frequently takes a large number of terms to approximate the function accurately. This is due to the fact that the periodic extension of the function is usually not very smooth. In this section we will introduce a superior form of interpolation in terms of Chebyshev polynomials. Suppose that f is an infinitely smooth function defined on the interval $[-1, 1]$. If we make the change of variable $x = \cos\theta$, then the function ϕ defined by

$$\phi(\theta) = f(\cos\theta)$$

is an infinitely smooth even periodic function on the whole real line. Therefore, from our results on trigonometric interpolation, we see that ϕ can be approximated by the interpolation function $\bar{\phi}$ given by

$$\bar{\phi}(\theta) = \sum_{n=0}^{N} a_n \cos(n\theta) \tag{5.103}$$

where the coefficients a_n are given by

$$a_0 = \frac{1}{N}\left[\tfrac{1}{2}(\phi_0 + \phi_N) + \sum_{m=1}^{N-1} \phi_m\right] \tag{5.104a}$$

$$a_N = \frac{1}{N}\left\{\tfrac{1}{2}\left[\phi_0 + (-1)^N \phi_N\right] + \sum_{m=1}^{N-1}(-1)^m \phi_m\right\} \tag{5.104b}$$

$$a_n = \frac{2}{N}\left\{\tfrac{1}{2}\left[\phi_0 + (-1)^n \phi_N\right] + \sum_{m=1}^{N-1} \phi_m \cos(\pi m n / N)\right\} \qquad n = 1, \ldots, N-1. \tag{5.104c}$$

Here ϕ_k is the value of ϕ at the point $\theta_k = k\pi/N$. Let us now interpret this interpolation result in terms of the original variable x. It follows that f can be interpolated by the function $\bar{f}$ defined by

$$\bar{f}(x) = \sum_{n=0}^{N} a_n T_n(x) \tag{5.105}$$

where the coefficients a_n are given by

$$a_0 = \frac{1}{N}\left[\frac{1}{2}(f_0 + f_N) + \sum_{m=1}^{N-1} f_m\right] \tag{5.106a}$$

$$a_N = \frac{1}{N}\left\{\frac{1}{2}\left[f_0 + (-1)^N f_N\right] + \sum_{m=1}^{N-1}(-1)^m f_m\right\} \tag{5.106b}$$

$$a_n = \frac{2}{N}\left\{\frac{1}{2}\left[f_0 + (-1)^n f_N\right] + \sum_{m=1}^{N-1} f_m T_m\big(\cos(\pi n/N)\big)\right\} \qquad n = 1, \ldots, N-1. \tag{5.106c}$$

Here f_k is the value of f at the point $x_k = \cos(k\pi/N)$. Notice that the interpolation points x_k are not uniformly spaced in the interval $[-1, 1]$. They are more dense near the endpoints of the interval. This nonuniform distribution of the interpolation points has the beneficial effect of increasing the convergence rate of the interpolation and providing a more uniform error distribution.

6 The Fast Fourier Transform (FFT)

A straight forward calculation of the discrete Fourier transform would take N^2 multiplications and additions. We will show that there are fast algorithms that can compute the discrete Fourier transform in $N \log_2 N$ operations. These algorithms are called FFT (Fast Fourier Transform) algorithms. The difference between N^2 and $N \log_2 N$ can be very significant for large N. For example, the savings for $N = 2048$ is a factor of 186. There are a number of different FFT algorithms. The one that will be described here is the one introduced by Danielson and Lanczos [4]. The FFT has made the widespread use of Fourier analysis possible.

6.1 Danielson-Lanczos Algorithm

This algorithm breaks the DFT into two smaller DFT's of half the size as shown below

$$
X_n = \sum_{m=0}^{N-1} x_m \, e^{i2\pi mn/N} = \sum_{m=0}^{N/2-1} x_{2m} \, e^{i2\pi(2m)n/N} + \sum_{m=0}^{N/2-1} x_{2m+1} \, e^{i2\pi(2m+1)n/N}
$$

$$
= \sum_{m=0}^{N/2-1} x_{2m} \, e^{i2\pi mn/(N/2)} + e^{i2\pi n/N} \sum_{m=0}^{N/2-1} x_{2m+1} \, e^{i2\pi mn/(N/2)} = X_n^e + e^{i2\pi n/N} X_n^o. \quad (6.1)
$$

In this equation X_n^e and X_n^o are the $N/2$ point transforms of the even and odd samples respectively. The index n in equation (6.1) runs from 0 to $N-1$, but the $N/2$ point transforms X_n^e and X_n^o are periodic with period $N/2$. This equation can be written as the pair of equations

$$
X_n = X_n^e + e^{i2\pi n/N} X_n^o \qquad n = 0, 1, \ldots, N/2 - 1 \qquad (6.2)
$$
$$
X_{n+N/2} = X_n^e - e^{i2\pi n/N} X_n^o \qquad n = 0, 1, \ldots, N/2 - 1. \qquad (6.3)
$$

If N is a power of 2, then X_n^e and X_n^o can be as expressed in terms of $N/4$ point transforms of their even and odd samples and this process can be repeated recursively until we arrive at one point transforms. The one point transform is just the identity operator. Thus, we wind up with the original samples in a different order. This order is called bit reversed order since it amounts to replacing the original indices in binary form by the indices formed by reversing the order of the bits. The recursive process is illustrated in figure 6.1.

Here we show the indices of the elements chosen at each stage along with their binary representation. The bit reversal occurs because choosing even and odd samples at the k-th step depends on whether the k-th bit from the right is a 1 or a 0. At each stage of this process there are N operations and there are $\log_2 N$ stages.

8			4			2			1	
0	000		0	000		0	000		0	000
1	001		2	010		4	100		4	100
2	010		4	100		2	010		2	010
3	011		6	110		6	110		6	110
4	100		1	001		1	001		1	001
5	101		3	011		5	101		5	101
6	110		5	101		3	011		3	011
7	111		7	111		7	111		7	111

Figure 6.1: Illustration of the bit reversal process for $N = 8$.

6.2 FFT of a Single Real Sequence

Let $x_0, x_1, \ldots, x_{N-1}$ be a sequence of real numbers. The DFT of this sequence could be obtained by defining a complex sequence with the values x_n in the real part and zero in the imaginary part and then applying the ordinary FFT to this complex sequence. However, in this section we will develop a more economical process.

Notice that the transformed sequence $X_0, X_2, \ldots, X_{N-1}$ has the property that the $N - k$ element is the conjugate of the k element. Define a new sequence of length $N/2$ by

$$h_m = x_{2m} + i\, x_{2m+1} \qquad m = 0, 1, \ldots, N/2 - 1. \tag{6.4}$$

Let $\{H_n\}$ denote the DFT of $\{h_m\}$. By linearity,

$$H_n = X_n^e + i\, X_n^o \qquad n = 0, 1, \ldots, N/2 - 1 \tag{6.5}$$

where X_n^e and X_n^o are the DFT's of $\{x_{2m}\}$ and $\{x_{2m+1}\}$. Since X_0^e and X_0^o are real, they can be obtained by taking the real and imaginary parts of H_0. Moreover,

$$H_{N/2-n} = X_{N/2-n}^e + i\, X_{N/2-n}^o = \overline{X_n^e} + i\, \overline{X_n^o}$$

or

$$\overline{H}_{N/2-n} = X_n^e - i\, X_n^0 \qquad n = 1, \ldots, N/2 - 1. \tag{6.6}$$

84

It follows from equations (6.5) and (6.6) that

$$X_n^e = \frac{1}{2}(H_n + \overline{H}_{N/2-n}) \tag{6.7}$$

$$X_n^o = \frac{1}{2i}(H_n - \overline{H}_{N/2-n}) \qquad n = 1, \ldots, N/2 - 1. \tag{6.8}$$

It follows from equations (6.2) and (6.3) that

$$X_n = X_n^e + e^{i2\pi n/N} X_n^o \qquad n = 0, 1, \ldots, N/2 - 1. \tag{6.9}$$

and

$$X_{N/2+n} = X_n^e - e^{i2\pi n/N} X_n^o \qquad n = 0, 1, \ldots, N/2 - 1. \tag{6.10}$$

The procedure can be summarized as follows:

1. Form the sequence $h_0, \ldots, h_{N/2-1}$ as in equation (6.4).

2. Take the FFT of $\{h_m\}$ to get $\{H_n\}$.

3. Compute X_n^e and X_n^o for $n = 0, \ldots, N/2 - 1$ using equations (6.7)and (6.8).

4. Compute X_n for $n = 0, \ldots, N - 1$ using equations (6.9) and (6.10).

6.3 Fast Discrete Sine Transform of Real Data

We have seen that Fourier sine series converge faster than the ordinary Fourier series when the function vanishes at the two endpoints, having a convergence rate of at least $1/n^3$. In this section we will show how to compute the discrete sine transform.

A real sequence $x_0, x_1, \ldots, x_{N-1}$ with $x_0 = 0$ can be extended to twice its length as an odd function as follows

$$x_N = 0 \quad and \quad x_{2N-m} = -x_m \qquad m = 1, 2, \ldots, N - 1. \tag{6.11}$$

The discrete Fourier transform of this odd sequence is given by

$$X_n = \sum_{m=0}^{2N-1} x_m e^{-\pi mn/N}$$

$$= \sum_{m=0}^{N-1} x_m e^{-i\pi mn/N} + \sum_{m=N+1}^{2N-1} x_m e^{-i\pi mn/N}$$

$$= \sum_{m=0}^{N-1} x_m e^{-i\pi mn/N} - \sum_{m=1}^{N-1} x_m e^{i\pi mn/N}$$

$$= -2i \sum_{m=0}^{N-1} x_m \sin(\pi mn/N) \qquad n = 0, 1, \ldots, 2N-1. \tag{6.12}$$

Notice that $X_N = 0$. We define the discrete sine transform by

$$\hat{X}_n = \sum_{m=0}^{N-1} x_m \sin(\pi mn/N) \qquad n = 0, 1, \ldots, N-1. \tag{6.13}$$

Notice that the values $\hat{X}_n$ of the discrete sine transform are real for all n. It follows from equation (6.12) that

$$X_{2N-n} = 2i \sum_{m=0}^{N-1} x_m \sin(\pi mn/N) \qquad n = 1, 2, \ldots, N. \tag{6.14}$$

Thus,

$$X_n = -2i\,\hat{X}_n \qquad n = 0, 1, \ldots, N-1 \tag{6.15a}$$

$$X_{2N-n} = +2i\,\hat{X}_n \qquad n = 1, 2, \ldots, N. \tag{6.15b}$$

Using the relations in equations (6.15a) and (6.15b), the inverse transform of X_n becomes

$$x_m = \frac{1}{2N} \sum_{n=0}^{2N-1} X_n e^{i\pi mn/N}$$

$$= \frac{-i}{N} \sum_{n=0}^{N-1} \hat{X}_n e^{i\pi mn/N} + \frac{1}{2N} \sum_{n=N+1}^{2N-1} X_n e^{i\pi mn/N}$$

$$= \frac{-i}{N} \sum_{n=0}^{N-1} \hat{X}_n e^{i\pi mn/N} + \frac{1}{2N} \sum_{n=1}^{N-1} X_{2N-n} e^{-i\pi mn/N}$$

$$= \frac{-i}{N} \sum_{n=0}^{N-1} \hat{X}_n e^{i\pi mn/N} + \frac{i}{N} \sum_{n=1}^{N-1} \hat{X}_n e^{-i\pi mn/N}$$

$$= \frac{2}{N} \sum_{n=0}^{N-1} \hat{X}_n \sin(\pi mn/N) \qquad m = 0, 1, \ldots, 2N-1. \qquad (6.16)$$

Therefore, the inverse discrete sine transform can be defined by

$$x_m = \frac{2}{N} \sum_{n=0}^{N-1} \hat{X}_n \sin(\pi mn/N) \qquad m = 0, 1, \ldots, N-1. \qquad (6.17)$$

Except for the factor $2/N$, the discrete sine transform and its inverse have the same form. Therefore, the same algorithm can be used for both. Now let us examine an economical way to calculate the discrete sine transform.

Define a new sequence $y_0, \ldots, y_{N-1}$ by

$$y_0 = 0 \quad and \quad y_m = (x_m + x_{N-m}) \sin(m\pi/N) + \tfrac{1}{2}(x_m - x_{N-m}) \qquad m = 1, \ldots, N-1 \quad (6.18)$$

and let R_m and I_m be the real and imaginary parts of the discrete Fourier transform of $\{y_m\}$. Since

$$\sum_{m=1}^{N-1} x_{N-m} \sin(m\pi/N) \cos(2\pi mn/N) = \sum_{m=1}^{N-1} x_m \sin(\pi - m\pi/N) \cos(\cos(2\pi n - 2\pi mn/N)$$

$$= \sum_{m=1}^{N-1} x_m \sin(m\pi/N) \cos(2\pi mn/N) \qquad (6.19)$$

and

$$\sum_{m=1}^{N-1} x_{N-m} \cos(2\pi mn/N) = \sum_{m=1}^{N-1} x_m \cos(2\pi n - 2\pi mn/N)$$

$$= \sum_{m=1}^{N-1} x_m \cos(2\pi mn/N), \tag{6.20}$$

it follows that

$$\begin{aligned}
R_n &= \sum_{m=0}^{N-1} y_m \cos(2\pi mn/N) \\
&= \sum_{m=1}^{N-1} (x_m + x_{N-m}) \sin(m\pi/N) \cos(2\pi mn/N) + \tfrac{1}{2} \sum_{m=1}^{N-1} (x_m - x_{N-m}) \cos(2\pi mn/N) \\
&= 2 \sum_{m=1}^{N-1} x_m \sin(m\pi/N) \cos(2\pi mn/N) + 0 \\
&= \sum_{m=1}^{N-1} x_m \left[\sin \frac{(2n+1)m\pi}{N} - \sin \frac{(2n-1)m\pi}{N} \right] \\
&= \hat{X}_{2n+1} - \hat{X}_{2n-1}. \tag{6.21}
\end{aligned}$$

Since

$$\sum_{m=1}^{N-1} x_{N-m} \sin(m\pi/N) \sin(2\pi mn/N) = \sum_{m=1}^{N-1} x_m \sin(\pi - m\pi/N) \sin(2\pi n - 2\pi mn/N)$$

$$= - \sum_{m=1}^{N-1} x_m \sin(m\pi/N) \sin(2\pi mn/N) \tag{6.22}$$

and

$$\sum_{m=1}^{N-1} x_{N-m} \sin(2\pi mn/N) = \sum_{m=1}^{N-1} x_m \sin(2\pi n - 2\pi mn/N)$$

$$= - \sum_{m=1}^{N-1} x_m \sin(2\pi mn/N), \tag{6.23}$$

it follows that

88

$$I_n = -\sum_{m=0}^{N-1} y_m \sin(2\pi mn/N)$$

$$= -\sum_{m=1}^{N-1} (x_m + x_{N-m}) \sin(m\pi/N) \sin(2\pi mn/N) - \tfrac{1}{2} \sum_{m=1}^{N-1} (x_m - x_{N-m}) \sin(2\pi mn/N)$$

$$= 0 - \sum_{m=1}^{N-1} x_m \sin(2\pi mn/N)$$

$$= -\hat{X}_{2n}. \tag{6.24}$$

Moreover,

$$R_0 = 2 \sum_{m=1}^{N-1} x_m \sin(m\pi/N) = 2\hat{X}_1. \tag{6.25}$$

Thus, the even terms of $\{\hat{X}_n\}$ can be computed using equation (6.24) and the odd terms can be computed recursively using equation (6.21) starting from $\hat{X}_1 = R_0/2$. The procedure can be summarized as follows:

1. Form the sequence $y_0, y_1, \ldots, y_{N-1}$ as in equation (6.18).

2. Take the FFT of the real sequence $\{y_n\}$ to obtain $\{R_m + i I_m\}$.

3. Compute the even terms of $\hat{X}_n$ using $\hat{X}_{2n} = -I_n$.

4. Compute the odd terms of $\hat{X}_n$ using the recursion relation $\hat{X}_{2n+1} = R_n + \hat{X}_{2n-1}$ starting with $\hat{X}_1 = R_0/2$.

6.4 Fast Discrete Cosine Transform of Real Data

The Fourier cosine series will often converge faster than the ordinary Fourier series, having a convergence rate of at least $1/n^2$. A two-dimensional Discrete Cosine Transform of sub blocks of pixels was used in the original JPEG image compression scheme. In this section we will show how to compute the Discrete Cosine Transform.

A real sequence $x_0, x_1, \ldots, x_N$ can be extended evenly to a sequence of $2N$ terms using

$$x_{2N-m} = x_m \qquad m = 1, 2, \ldots, N-1. \tag{6.26}$$

The discrete Fourier transform of this even sequence can be written

$$X_n = \sum_{m=0}^{2N-1} x_m e^{-i\pi mn/N}$$

$$= \left[x_0 + (-1)^n x_N\right] + \sum_{m=1}^{N-1} x_m e^{-i\pi mn/N} + \sum_{m=N+1}^{2N-1} x_m e^{-i\pi mn/N}$$

$$= \left[x_0 + (-1)^n x_N\right] + \sum_{m=1}^{N-1} x_m e^{-i\pi mn/N} + \sum_{m=1}^{N-1} x_{2N-m} e^{i\pi mn/N}$$

$$= \left[x_0 + (-1)^n x_N\right] + 2\sum_{m=1}^{N-1} x_m \cos(\pi mn/N) \qquad n = 0, 1, \ldots, 2N-1. \qquad (6.27)$$

We define the discrete cosine transform by

$$\hat{X}_n = \tfrac{1}{2}X_n = \tfrac{1}{2}\left[x_0 + (-1)^n x_N\right] + \sum_{m=1}^{N-1} x_m \cos(\pi mn/N) \qquad n = 0, 1, \ldots, N. \qquad (6.28)$$

It follows from equation(6.27) that

$$X_{2N-n} = \left[x_0 + (-1)^n x_N\right] + 2\sum_{m=1}^{N-1} x_m \cos(\pi mn/N) \qquad n = 1, 2, \ldots, N-1. \qquad (6.29)$$

Thus,

$$X_n = 2\hat{X}_n \qquad n = 0, 1, \ldots, N \qquad (6.30a)$$

$$X_{2N-n} = 2\hat{X}_n \qquad n = 1, 2, \ldots, N-1. \qquad (6.30b)$$

Using the relations in equations (6.30a) and (6.30b), the inverse transform of X_n becomes

$$x_m = \frac{1}{2N} \sum_{n=0}^{2N-1} X_n e^{i\pi mn/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N} \hat{X}_n e^{i\pi mn/N} + \frac{1}{2N} \sum_{n=N+1}^{2N-1} X_n e^{i\pi mn/N}$$

$$= \frac{1}{N}\left[\hat{X}_0 + (-1)^m \hat{X}_N\right] + \frac{1}{N} \sum_{n=1}^{N-1} \hat{X}_n e^{i\pi mn/N} + \frac{1}{2N} \sum_{n=1}^{N-1} X_{2N-n} e^{-i\pi mn/N}$$

$$= \frac{1}{N}\left[\hat{X}_0 + (-1)^m \hat{X}_N\right] + \frac{1}{N} \sum_{n=1}^{N-1} \hat{X}_n e^{i\pi mn/N} + \frac{1}{N} \sum_{n=1}^{N-1} \hat{X}_n e^{-i\pi mn/N}$$

$$= \frac{1}{N}\left\{\left[\hat{X}_0 + (-1)^m \hat{X}_N\right] + 2 \sum_{n=1}^{N-1} \hat{X}_n \cos(\pi mn/N)\right\}$$

$$= \frac{2}{N}\left\{\tfrac{1}{2}\left[\hat{X}_0 + (-1)^m \hat{X}_N\right] + \sum_{n=1}^{N-1} \hat{X}_n \cos(\pi mn/N)\right\} \qquad m = 0, 1, \dots, 2N - 1. \quad (6.31)$$

Therefore, we can define the inverse discrete cosine transform by

$$x_m = \frac{2}{N}\left\{\tfrac{1}{2}\left[\hat{X}_0 + (-1)^m \hat{X}_N\right] + \sum_{n=1}^{N-1} \hat{X}_n \cos(\pi mn/N)\right\} \qquad m = 0, 1, \dots, N. \quad (6.32)$$

Again the inverse discrete cosine transform has the same form as the discrete cosine transform except for a factor of $2/N$. Thus the same algorithm can be used for both the transform and its inverse.

Like the sine transform, the cosine transform can be computed from the discrete Fourier transform of an auxiliary sequence $\{y_m\}$ defined by

$$y_m = \tfrac{1}{2}(x_m + x_{N-m}) - (x_m - x_{N-m})\sin(m\pi/N) \qquad m = 0, 1, \dots, N - 1. \quad (6.33)$$

Let R_n and I_n be the real and imaginary parts of the discrete Fourier transform of the sequence $\{y_m\}$. It follows from equations (6.19) and (6.20) that

$$R_n = \sum_{m=0}^{N-1} y_m \cos 2\pi mn/N$$

$$= \tfrac{1}{2} \sum_{m=0}^{N-1} (x_m + x_{N-m}) \cos(2\pi mn/N) - \sum_{m=0}^{N-1} (x_m - x_{N-m}) \sin(m\pi/N) \cos(2\pi mn/N)$$

$$= \tfrac{1}{2}(x_0 + x_N) + \sum_{m=1}^{N-1} \cos(2\pi mn/N) - 0 = \hat{X}_{2n} \qquad n = 0, 1, \dots, N/2. \qquad (6.34)$$

It follows from equations (6.22) and (6.23) that

$$I_n = -\sum_{m=0}^{N-1} y_m \sin 2\pi mn/N$$

$$= -\tfrac{1}{2} \sum_{m=0}^{N-1} (x_m + x_{N-m}) \sin(2\pi mn/N) + \sum_{m=0}^{N-1} (x_m - x_{N-m}) \sin(m\pi/N) \sin(2\pi mn/N)$$

$$= -0 + 2 \sum_{m=1}^{N-1} \sin(m\pi/N) \sin(2\pi mn/N)$$

$$= \sum_{m=1}^{N-1} \left[\cos\big(\pi m(2n - 1)/N\big) - \cos\big(\pi m(2n + 1)/N\big) \right]$$

$$= \hat{X}_{2n-1} - \hat{X}_{2n+1} \qquad n = 1, 2, \dots, N/2 - 1. \qquad (6.35)$$

Thus the even terms of $\{\hat{X}_n\}$ can be computed using equation (6.34) and the odd terms can be computed recursively using equation (6.35). The starting value for the recursion $\hat{X}_1$ can be computed from its definition

$$\hat{X}_1 = \tfrac{1}{2}(x_0 - x_N) + \sum_{m=1}^{N-1} x_m \cos(\pi m/N). \qquad (6.36)$$

The procedure can be summarized as follows:

1. Form the sequence $\{y_m\}$ using equation (6.33)

2. Take the FFT of the real sequence $\{y_m\}$ to obtain $\{R_n + i I_n\}$.

3. Compute the even terms $\hat{X}_{2n}$ using $\hat{X}_{2n} = R_n$.

4. Compute the odd terms $\hat{X}_{2n+1}$ using the recursion $\hat{X}_{2n+1} = \hat{X}_{2n-1} - I_n$ starting from $\hat{X}_1 = \tfrac{1}{2}(x_0 - x_N) + \sum_{m=1}^{N-1} x_m \cos(\pi m/N)$.

References

[1] Clenshaw, C.W., *The Numerical Solution of Linear Differential Equations in Chebyshev Series*, Proc. Cambridge Philosophical Society, **53**, pp. 134–149, 1957.

[2] Clenshaw, C.W., Mathematical Tables, Vol. 5, National Physical Laboratory, 1962.

[3] Cooley, J.W. and Tukey, J.W., *An Algorithm for the Machine Calculation of Complex Fourier Series*, Mathematics of Computation, **19**, 297,1965.

[4] Danielson, G.C. and Lanczos, Lanczos, C., *Some Improvements in Practical Fourier Analysis and Their Applications to X-ray Scattering from Liquids*, Journal of the Franklin Institute, **233**, No. 4, 365–452, 1942.

[5] Fox, L., *Chebyshev Methods for Ordinary Differential Equations*, The Computer Journal, **4**, no. 4, 318–331, 1962.

[6] Gauss, C.F., *Nachlass: Theoria interpolations methodo nova tractata*, in Carl Fredrich Gauss, Werke, Band 3, Göttingen Königlichen Gesellshaft der Wissenschaften, 265–303, 1866.

[7] Lanczos, C., *Trigonometric Interpolation of Empirical and Analytical Functions*, Journal of Mathematics and Physics, **17**, 123–199, 1938.

[8] Lanczos, C., Applied Analysis, Prentice Hall, 1956.

[9] Ortiz, E.L., *The Tau Method*, SIAM J. Numer. Anal., **6**, No. 3, 480–492, 1969.

[10] Ortiz, E.L. and Samara, H., *An Operational Approach to the Tau Method for the Numerical Solution of Non-Linear Differential Equations*, Computing, **27**, 15–25, 1981.

[11] Press, W.H., Flannery, B.P., Teukolsky, S.A., and Vetterling, W.T., Numerical Recipes in Fortran, Cambridge University Press, 1992.

[12] Benthien, G.W., *Acoustic Array Interactions in the Time Domain*, A longer version of the paper given at the Acoustics08 conference in Paris [a pdf version is on the website www.gbenthien.net/papers/index.html].