Mutual Interaction of Pistons of Arbitrary Shape on a Planar Rigid Baffle

Dr. George W Benthien

November 1985

E-mail: george@gbenthien.net

Oscar Lindeman in a 1974 JASA article [Lindeman, Oscar, *Transient fluid reaction on a baffled plane piston of arbitrary shape*, J. Acoust. Soc. Am., **55**, No. 4, April 1974] presented a technique for obtaining the self radiation impedance of a baffled piston on a plane as a Fourier transform of its impulse response. The impulse response was expressed as a double contour integral around the pistons perimeter. This technique has been applied by a number of authors to pistons of various shapes. In this paper I will illustrate how Lindeman's technique can be extended to the mutual acoustic interaction between two pistons of arbitrary shape on a rigid planar baffle. We will denote the two pistons by $\mathcal{P}$ and $\mathcal{P}'$ as shown in figure 1.

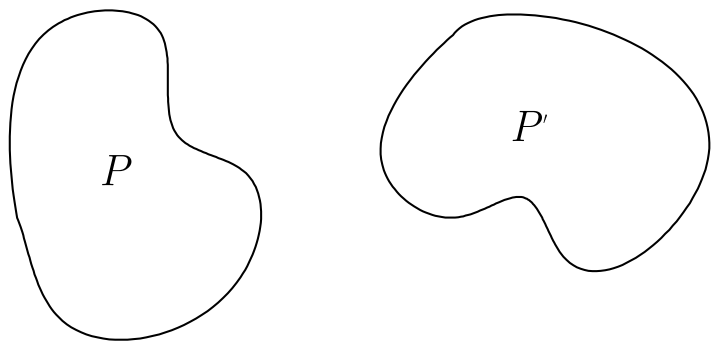

Figure 1: Two planar pistons of arbitrary shape on an infinite rigid baffle

The mutual radiation impedance $\mathcal{Z}$ is defined by

$$\mathcal{Z} = \rho c \frac{ik}{2\pi} \int_P \int_{P'} \frac{e^{-ik|x-x'|}}{|x-x'|} \, dS(x) dS(x'). \tag{1}$$

We will show that the double surface integral in equation (1) can be reduced to a simpler expression involving only line integrals. The technique used is similar to that used by Lindeman to calculate self radiation reactions [Lindeman,Oscar,*Transformation of the Helmholtz integral into a Line Integral*, J. Acoust. Soc. Am., **40**, no. 4, 1966]. It relies on the two dimensional form of Green's theorem.

By Green's theorem in two dimensions

$$\int_P \nabla_x \, \phi(x, x') \, dS(x) = \oint_{\partial P} \phi(x, x') \vec{n}(x) \, ds(x) \tag{2}$$

where ∂P is the boundary of P and $\vec{n}$ is the outward unit normal to ∂P. By the divergence theorem in two dimensions

$$\int_{P'} \operatorname{div} \vec{u}(x') \, dS(x') = \oint_{\partial P'} \vec{u}(x') \cdot \vec{n}(x') \, ds(x'). \tag{3}$$

It follows from equations (2) and (3) that

$$\int_{P'} \mathrm{div}_{x'} \left[\int_P \nabla_x \, \phi(x,x') \, dS(x) \right] dS(x') = \oint_{\partial P'} \vec{n}(x') \cdot \left[\int_P \nabla_x \, \phi(x,x') \, dS(x) \right] ds(x')$$
$$= \oint_{\partial P'} \vec{n}(x') \cdot \left[\oint_{\partial P} \phi(x,x') \vec{n}(x) \, ds(x) \right] ds(x'). \quad (4)$$

Suppose ϕ has the form

$$\phi(x,x') = \Phi(x-x'). \quad (5)$$

Then

$$\nabla_x \, \phi(x,x') = -\nabla_{x'} \, \phi(x,x') \quad (6)$$

and

$$\triangle_x \, \phi(x,x') = \triangle_{x'} \, \phi(x,x'). \quad (7)$$

Combining equations (4)–(7), we get

$$\oint_{\partial P'} \vec{n}(x') \cdot \left[\oint_{\partial P} \phi(x,x') \vec{n}(x) \, ds(x) \right] ds(x') = - \int_{P'} \mathrm{div}_{x'} \left[\int_P \nabla_{x'} \, \phi(x,x') \, dS(x) \right] dS(x')$$
$$= \int_{P'} \left[\int_P \triangle_{x'} \, \phi(x,x') \, dS(x) \right] dS(x')$$
$$= \int_{P'} \left[\int_P \triangle_x \, \phi(x,x') \, dS(x) \right] dS(x'). \quad (8)$$

If ϕ has the form

$$\phi(x,x') = \psi(r) \quad \text{with} \quad r = |x-x'|, \quad (9)$$

then

$$\triangle_x \, \phi(x,x') = \frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right). \quad (10)$$

Combining equations (8) and (10), we get

$$\oint_{\partial P'} \vec{n}(x') \cdot \left[\oint_{\partial P} \phi(x,x') \vec{n}(x) \, ds(x) \right] ds(x') = - \int_{P'} dS(x') \int_P \frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) dS(x). \quad (11)$$

If we define

$$\psi(r) = \int_0^r \frac{1 - e^{-ikz}}{z} \, dz, \quad (12)$$

then

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) = ik \, \frac{e^{-ikr}}{r}. \quad (13)$$

Combining equations (11)–(13), we get

$$
\begin{aligned}
ik \int_{P'} dS(x') \int_P \frac{e^{-ik|x-x'|}}{|x-x'|}\, dS(x) &= -\oint_{\partial P'} \vec{n}(x') \cdot \left[\oint_{\partial P} \psi(|x-x'|) \vec{n}(x)\, ds(x) \right] ds(x') \\
&= \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \int_0^{|x-x'|} \frac{1-e^{-ikz}}{z}\, dz\, ds(x) ds(x').
\end{aligned}
\tag{14}
$$

If we define $\chi(x, x', z)$ by

$$
\chi(x, x', z) = \begin{cases} 1 & |x-x'| \geq z \\ 0 & |x-x'| < z \end{cases} \tag{15}
$$

then

$$
\int_0^{|x-x'|} \frac{1-e^{-ikz}}{z}\, dz = \int_0^w \chi(x, x', z) \frac{1-e^{-ikz}}{z}\, dz \tag{16}
$$

where

$$
w = \max_{\substack{x \in P \\ x' \in P'}} |x-x'|. \tag{17}
$$

Substituting equation (16) into equation (14), we get

$$
\begin{aligned}
ik \int_{P'} dS(x') \int_P \frac{e^{-ik|x-x'|}}{|x-x'|}\, dS(x) &= -\oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \int_0^w \chi(x, x', z) \frac{1-e^{-ikz}}{z}\, dz \\
&= 2\pi \int_0^w f(z)(1-e^{-ikz})\, dz
\end{aligned}
\tag{18}
$$

where

$$
f(z) = -\frac{1}{2\pi z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \chi(x, x', z)\, ds(x) ds(x'). \tag{19}
$$

Substituting equation (18) into equation (1), we get

$$
\mathcal{Z} = \rho c \int_0^w f(z)(1-e^{-ikz})\, dz. \tag{20}
$$

It is shown in the appendix that

$$
\int_0^w f(z)\, dz = \text{Area}(P \cap P'). \tag{21}
$$

Combining equations (20) and (21), we get

$$
\boxed{\mathcal{Z} = \rho c\, \text{Area}(P \cap P') - \rho c \int_0^w f(z) e^{-ikz}\, dz.} \tag{22}
$$

Letting $t = z/c$ in the above integral, we obtain

$$\mathcal{Z}(\omega) = \rho c \,\text{Area}(P \cap P') - \rho c^2 \int_0^{w/c} f(ct) e^{-i\omega t}\, dt. \tag{23}$$

The radiation impulse response $\zeta(t)$ of the piston is the inverse Fourier transform of $\mathcal{Z}(\omega)$, i.e.,

$$\zeta(t) = \rho c \,\text{Area}(P \cap P')\, \delta(t) - \rho c^2 f(ct) \tag{24}$$

where $\delta(t)$ is the Dirac delta function.

Since $\oint_{\partial P} \vec{n}(x)\, ds(x) = 0$, f can be written in the alternate form

$$\begin{aligned} f(z) &= \frac{1}{2\pi z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x)[1 - \chi(x, x', z)]\, ds(x) ds(x') \\ &= \frac{1}{2\pi z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \hat{\chi}(x, x', z)\, ds(x) ds(x') \end{aligned} \tag{25}$$

where

$$\hat{\chi}(x, x', z) = 1 - \chi(x, x', z) = \begin{cases} 1 & |x - x'| < z \\ 0 & |x - x'| \geq z \end{cases}. \tag{26}$$

If we define

$$g(z, x') = \oint_{\partial P} \vec{n}(x') \cdot \vec{n}(x) \hat{\chi}(x, x', z)\, ds(x), \tag{27}$$

then

$$f(z) = \frac{1}{2\pi z} \oint_{\partial P'} g(z, x')\, ds(x'). \tag{28}$$

Geometrically, $g(z, x')$ is the signed projection of the portion of ∂P within a circle of radius z centered at x' onto the tangent line at x'. For example, $g(z, x')$ is the sum of the lengths of the segments ab and cd in figure 2 for the case $P = P'$. There is no contribution in the interval (b, c) since the contributions of the two portions of ∂P within the circle have opposite signs and cancel.

It is clear from the figure that, for small z, $g(z, x')$ is approximately $2z$. This is true for all points x' where ∂P is smooth. We will assume that the set of points where ∂P is not smooth is a set of measure zero. Therefore,

$$f(0^+) = \sigma/\pi \qquad \text{when} \qquad P = P' \tag{29}$$

where σ is the perimeter of P. When $P \cap P' = \emptyset$, there is a minimum distance between points on ∂P and points on $\partial P'$. If z is less than this minimum distance, then $g(z, x') = 0$ for all $x' \in \partial P'$. Therefore,

$$f(0^+) = 0 \qquad \text{when} \qquad P \cap P' = \emptyset. \tag{30}$$

It is also clear from the geometric interpretation of $g(z, x')$ that it is bounded by $2z$. Therefore, $f(z)$ is bounded by σ/π.

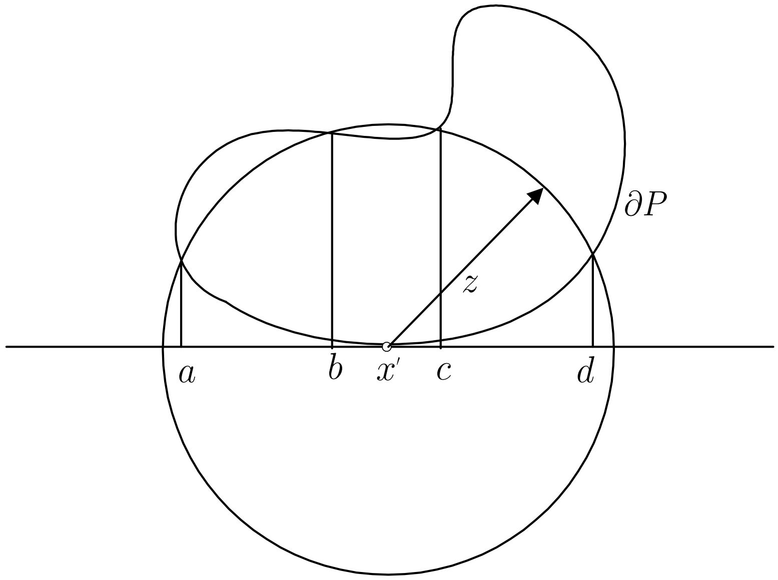

Figure 2: Geometric interpretetion of $g(z, x')$ for the case $P = P'$

Appendix

In this appendix we want to show that

$$\int_0^w f(z)\, dz = \text{Area}(P \cap P').$$

It was shown in equation (25) that

$$f(z) = \frac{1}{2\pi z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \hat{\chi}(x, x', z)\, ds(x) ds(x').$$

Therefore,

$$\begin{aligned} \int_0^w f(z)\, dz &= \frac{1}{2\pi} \int_0^w \frac{dz}{z} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \hat{\chi}(x, x', z)\, ds(x) ds(x') \\ &= \frac{1}{2\pi} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \int_{|x-x'|}^w \frac{dz}{z}\, ds(x) ds(x') \\ &= \frac{1}{2\pi} \oint_{\partial P'} \vec{n}(x') \cdot \oint_{\partial P} \vec{n}(x) \ln |x - x'|\, ds(x) ds(x'). \end{aligned} \tag{31}$$

In deriving equation (31) we used the fact that

$$\oint_{\partial P} \vec{n}(x)\, ds(x) = 0.$$

If $x' \notin P$, then it follows from Greeen's theorem that

$$\oint_{\partial P} \vec{n} \ln|x - x'|\, ds(x) = \int_P \nabla_x \ln|x - x'|\, dS(x). \tag{32}$$

Suppose $x' \in P$. Let $S_\epsilon(x')$ be a circle of radius ϵ centered at x' [see figure 3]. By Green's

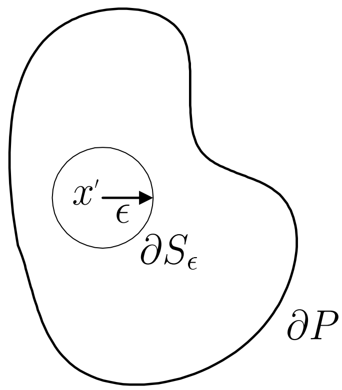

Figure 3: Isolation of singular point by a circle of radius ϵ

theorem

$$\oint_{\partial P + \partial S_\epsilon(x')} \vec{n} \ln|x - x'|\, ds(x) = \int_{P - S_\epsilon(x')} \nabla_x \ln|x - x'|\, dS(x). \tag{33}$$

But

$$\oint_{\partial S_\epsilon(x')} \vec{n} \ln|x - x'|\, ds(x) = \ln\epsilon \oint_{\partial S_\epsilon(x')} \vec{n}\, ds(x) = 0. \tag{34}$$

Thus,

$$\oint_{\partial P} \vec{n} \ln|x - x'|\, ds(x) = \lim_{\epsilon \to 0} \int_{P - S_\epsilon(x')} \nabla_x \ln|x - x'|\, dS(x) = \int_P \nabla_x \ln|x - x'|\, dS(x). \tag{35}$$

Combining equations (31)–(35), we get

$$\begin{aligned} \int_0^w f(z)\, dz &= \frac{1}{2\pi} \oint_{\partial P'} \vec{n}(x') \cdot \int_P \nabla_x \ln|x - x'|\, dS(x) ds(x') \\ &= -\frac{1}{2\pi} \oint_{\partial P'} \vec{n}(x') \cdot \int_P \nabla_{x'} \ln|x - x'|\, dS(x) ds(x') \\ &= -\frac{1}{2\pi} \int_P dS(x) \oint_{\partial P'} \nabla_{x'} \ln|x - x'| \cdot \vec{n}(x')\, ds(x'). \end{aligned} \tag{36}$$

If $x \notin P'$, then it follows from the divergence theorem that

$$\begin{aligned} \oint_{\partial P'} \nabla_{x'} \ln|x - x'| \cdot \vec{n}(x')\, ds(x') &= \int_{P'} \triangle_{x'} \ln|x - x'|\, dS(x') \\ &= 0. \end{aligned} \tag{37}$$

Thus,

$$\int_0^w f(z)\,dz = -\frac{1}{2\pi}\int_{P\cap P'} dS(x) \oint_{\partial P'} \nabla_{x'}\, \ln|x-x'|\cdot\vec{n}(x')\,ds(x'). \tag{38}$$

Let $S_\epsilon(x)$ be a circle around x of radius ϵ. Then, it follows from the divergence theorem that

$$\begin{aligned}\oint_{\partial P'+\partial S_\epsilon} \nabla_{x'}\, \ln|x-x'|\cdot\vec{n}(x')\,ds(x') &= \int_{P'-S_\epsilon} \triangle_{x'}\, \ln|x-x'|\,dS(x')\\ &= 0\end{aligned} \tag{39}$$

or

$$\oint_{\partial P'} \nabla_{x'}\, \ln|x-x'|\cdot\vec{n}(x')\,ds(x') = -\oint_{\partial S_\epsilon} \nabla_{x'}\, \ln|x-x'|\cdot\vec{n}(x')\,ds(x'). \tag{40}$$

However,

$$\begin{aligned}\oint_{\partial S_\epsilon} \nabla_{x'}\, \ln|x-x'|\cdot\vec{n}(x')\,ds(x') &= \oint_{\partial S_\epsilon} \frac{1}{|x-x'|}\nabla_{x'}\,|x-x'|\cdot\vec{n}(x')\,ds(x')\\ &= \oint_{\partial S_\epsilon} \frac{1}{|x-x'|}\frac{x'-x}{|x-x'|}\cdot\vec{n}(x')\,ds(x')\\ &= \oint_{\partial S_\epsilon} \frac{1}{|x-x'|}\,ds(x')\\ &= \frac{1}{\epsilon}\oint_{\partial S_\epsilon} ds(x')\\ &= 2\pi.\end{aligned} \tag{41}$$

Combining equations (40) and (41), we get

$$\oint_{\partial P'} \nabla_{x'}\, \ln|x-x'|\cdot\vec{n}(x')\,ds(x') = -2\pi. \tag{42}$$

Substituting equation (42) into equation (38), we get

$$\int_0^w f(z)\,dz = \int_{P\cap P'} dS(x) = \text{Area}(P\cap P'). \tag{43}$$

In particular, we have

$$\int_0^w f(z)\,dz = \begin{cases}\text{Area}(P) & \text{for } P=P'\\ 0 & \text{for } P\cap P'=\emptyset\end{cases}. \tag{44}$$